THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 60, No. 10, December 1981
Printed in U.S.A.

Conformal Mapping and Complex Coordinates in Cassegrainian and Gregorian Reflector Antennas

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(Manuscript received June 18, 1981)

In a Gregorian or Cassegrainian reflector antenna, the complex coordinate u' of an output ray is related to the corresponding input coordinate u by a bilinear transformation, $u' = (au + b)/(cu + d)$. We discuss the properties of this transformation, derive its coefficients a, b, c, d, and give explicitly the conditions that must be satisfied in order that symmetry be preserved. The conditions are expressed directly in terms of the parameters that specify the path of the principal ray, which is the ray corresponding to the feed axis. The results are directly related to well-known properties of stereographic projections, and they are shown to be useful in the design of multi-reflector antennas which minimize aberrations and cross-polarization.

I. INTRODUCTION

Gregorian and Cassegrainian reflector arrangements are needed for ground station and satellite antennas, and terrestrial radio relay systems.[1–8] In these antennas, a paraboloid of large aperture is combined with a smaller subreflector (an hyperboloid or an ellipsoid). The feed is placed at the antenna focal point, and it illuminates the subreflector with a spherical wave, which is then transformed by the two reflectors into a plane wave. Each input ray from the feed is thus transformed into an output ray parallel to the paraboloid axis.

This transformation can be represented by a stereographic projection.[9–11] Therefore, it is a conformal mapping—it transforms circles into circles, and it is described by the bilinear transformation

$$u' = \frac{au + b}{cu + d}, \tag{1}$$

where u is the "complex coordinate" of an input ray and u' the corresponding output coordinate. In this article, we discuss the properties of the bilinear transformation, derive its coefficients a, b, c, d, and give explicitly the conditions that must be satisfied to obtain circular symmetry, in which case

$$b = c = 0, \qquad d = 1. \tag{2}$$

The results are related to well-known properties of stereographic projections, and they generalize previous results in Refs. 12 to 18.

We first consider, in Section II, an ellipsoid illuminated by a spherical wave front S. We assume that S originates from one of the two foci of the ellipsoid, and determine the properties of a reflected wave front S', assuming geometric optics. We determine for each point P' of S' the corresponding point P of the incident wave front S and show that the correspondence $P \rightarrow P'$ is everywhere a conformal mapping. According to a well-known theorem of complex variables,[9] such a conformal mapping can be represented by the bilinear transformation (1), provided suitable complex variables u' and u are defined for the rays through P' and P. A suitable choice is obtained with two separate reference frames for P' and P using the familiar relations[9–13]

$$u = e^{j\phi}\tan\frac{\theta}{2} \qquad u' = e^{j\phi'}\tan\frac{\theta'}{2}, \tag{3}$$

where θ', ϕ' and θ, ϕ are spherical coordinates.

Since the two reference frames defining u and u' can be oriented arbitrarily, eq. (1) implies that an arbitrary rotation of the input frame must transform the input coordinate u according to a bilinear transformation.[10] The coefficients a, b, c, d of such a rotation are derived in Section V, where it is shown, for a rotation characterized by Euler angles α, β, γ, that

$$a = 1, \qquad b = -e^{j\alpha}\tan\frac{\beta}{2}, \tag{4}$$

$$c = e^{j\gamma}\tan\frac{\beta}{2}, \qquad d = e^{j(\alpha+\gamma)}. \tag{5}$$

Since an ellipsoid has an axis of symmetry, it is always possible to orient the input and output frames so as to reduce eq. (1) to the normal form

$$u' = au. \tag{6}$$

However, if the feed is centered around the z-axis of the input frame, then the reflected wave is blocked by the feed. For this reason, to avoid blockage, the z-axis must be tilted with respect to the ellipsoid axis.

Then, using eqs. (4) to (6) and properly orienting the input and output frames, it is shown in Section VI that eq. (1) assumes the form

$$u' = M \frac{u}{1 + (M - 1)u \tan i}, \tag{7}$$

where M and i are the magnification and angle of incidence for the principal ray corresponding to the feed axis (i.e., they ray $u = 0$).

The product of the two transformations given by eq. (6) and eqs. (4) to (5) gives the group of all possible transformations that can be obtained with an ellipsoid. One can show that this is the complete group of bilinear transformations. Thus, for any given values of the coefficients, a, b, c, d, it is always possible to find an ellipsoid (combined with suitable reference frames) which will produce the transformation (1) with the specified values of a, b, c, d. In Section VII, we consider an antenna consisting of N reflectors, each represented by a bilinear transformation. Obviously, the product of the N transformations is again a bilinear transformation and, therefore, the antenna can be represented by an equivalent ellipsoid.

Most antennas are focused at ∞. Then the equivalent ellipsoid becomes a paraboloid, and the antenna can be represented as shown in Fig. 1, showing a feed illuminating the equivalent paraboloid from its focus 0. The coefficients a, b, c, d in this case are obtained by letting $M \to 0$ in eq. (7), and it is shown in Section VI that we then obtain, for the complex coordinate $x' + jy'$ of an output ray,

$$x' + jy' = 2f \frac{u}{1 - u \tan i}, \tag{8}$$

where u is the input coordinate and $f = OI$ is the focal length of the equivalent paraboloid.

In Section VI, we derive a simple expression for the coefficient tan i in terms of the angles of incidence specifying the orientations of the various reflectors with respect to the principal ray. The value of tan i is needed in the design of a multibeam antenna to determine the aberrations caused by a small feed displacement from the focus. It is also needed to determine the output polarization, as shown in Section VIII.

Of special importance is the condition

$$\tan i = 0. \tag{9}$$

Then, using a corrugated feed[20] (or a feed with similar characteristics[21,22]) the output wave fronts become everywhere polarized in one direction. Furthermore, astigmatism is eliminated[19] for small feed displacements in a multibeam antenna. The above condition can

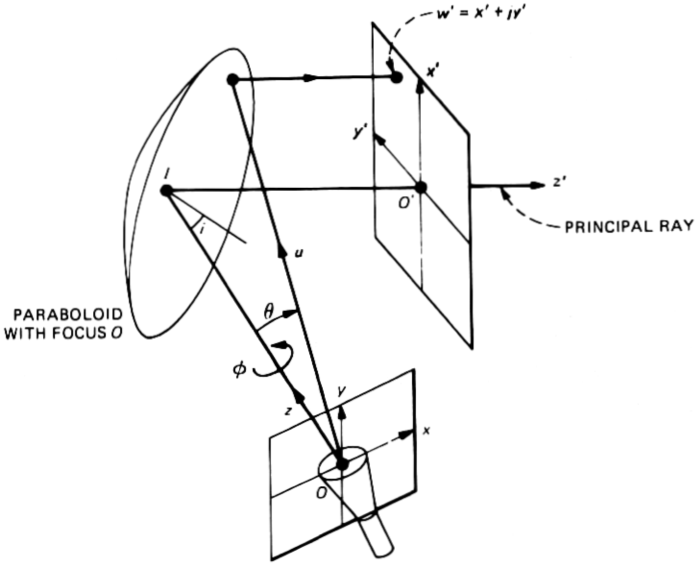

Fig. 1—A Cassegrainian or Gregorian reflector arrangement is represented by an equivalent paraboloid combined with a feed at 0. The principal ray corresponding to the feed axis is reflected at I with angle of incidence i.

always be satisfied by properly orienting the feed axis[15–17] as shown in Section VIII.

The above results are related to previous results by Brickell and Westcott,[13,14] Tanaka, Mizusawa,[15] Mizuguchi et al.,[16] Hanfling,[12] and the author.[17] There is a simple connection, pointed out in Section IX, between some of the expressions derived here and certain results in Refs. 13 and 14; the particular case, $N = 2$, is treated in those references. When only two reflectors are involved, there is always a common plane of symmetry, and the feed orientation can be derived geometrically as in Ref. 17. Our results differ from those of Refs. 15 and 16 in two respects: first, they apply also for $N > 2$; second, $\tan i$ is expressed directly in terms of the parameters (magnifications and angles of incidence) that specify the path of the principal ray. As pointed out in Ref. 19, an important application of our results is in the theory of aberrations.

The main results of this article are derived in Sections V through VII. Most of the results of Sections II through IV are well-known properties of ellipsoids, but their derivation is needed for Sections V through VII. In the following section, we discuss the transformation obtained when an ellipsoid is illuminated by a spherical wave front originating from one of the two foci. This transformation has the following basic property: it is a conformal mapping which gives correctly, not only the amplitude distribution of a reflected wave front, but also its polarization. All results of this article directly follow from this property.

II. CONFORMAL MAPPING, COMPLEX COORDINATES, AND THE BILINEAR TRANSFORMATION

Let a linearly polarized point source be placed at O, one of the two foci of an ellipsoid, and let O' be the other focus (in Fig. 2). Then for each ray from O, a corresponding ray through O' is obtained after reflection by the ellipsoid. To determine the properties of this correspondence, introduce at the two foci separate coordinate systems x, y,

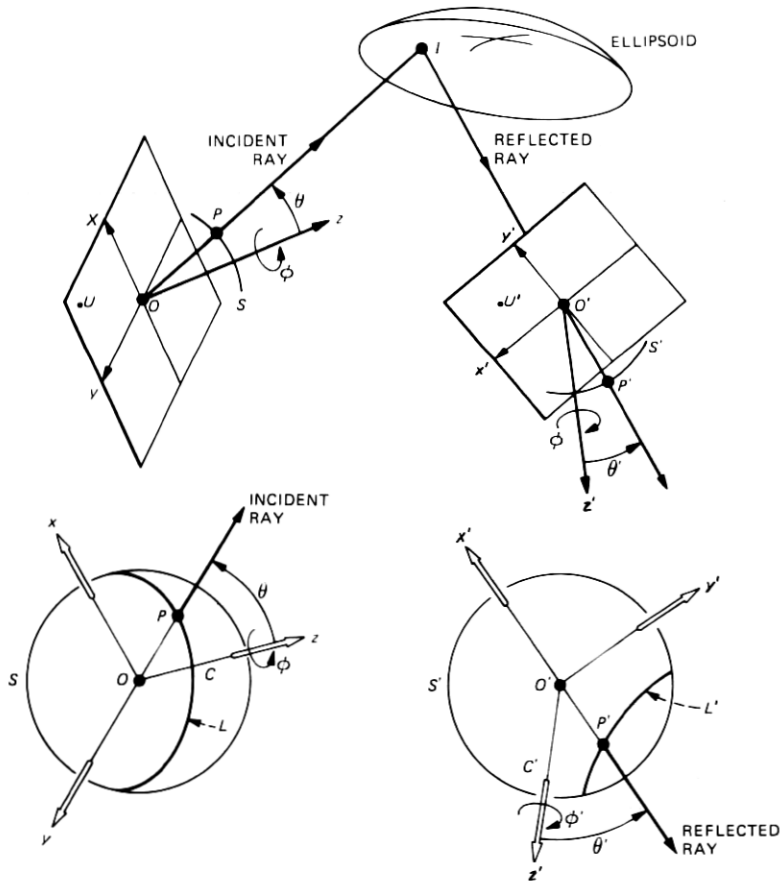

Fig. 2—A ray from one of the two foci of an ellipsoid determines, after reflection, a ray through the other focus. The correspondence $P \rightarrow P'$ between points of two wave fronts S and S' is everywhere as conformal mapping described by eq. (1).

z and x', y', z' oriented arbitrarily, as shown in Fig. 2. Consider an incident ray from O with spherical coordinates θ, ϕ, and let θ', ϕ' be the spherical coordinates of the corresponding ray through O'. It is convenient to introduce as in Refs. 12, 13 complex coordinates u and u' defined by eq. (3). Then, we show in this section that u' and u are related by a bilinear transformation. If both coordinate systems are right-handed, we shall see that the bilinear transformation does not relate u' directly to u, but to its complex conjugate u^*. To avoid this inconvenience, one of the two reference systems will be assumed to be left-handed as shown in Fig. 2.

To better visualize the one-to-one correspondence between rays through the two foci, consider in Fig. 2 two wave fronts S and S' centered at O and O', respectively. Then, for each ray from O, one obtains on S and S' two corresponding points P and P'. Furthermore, letting P be a variable point of a curve L of S, one obtains on S' a variable point P' describing a corresponding curve L' of S'. Since the point source is linearly polarized, the curve L can be drawn so that it is everywhere tangent to the magnetic field. Then one obtains a polarization line of S, and it is shown in Appendix A that the correspondence $P \rightarrow P'$ transforms polarization lines into polarization lines. That is,

if L is a polarization line,
then L′ is also a
polarization line. (10)

Another property is that angles are preserved and, therefore, the correspondence $P \rightarrow P'$ is a *conformal mapping*.

The above considerations apply also to an hyperboloid (in which case one of the two foci is behind the reflector), to a paraboloid, or to any combination of such reflectors. Thus, let the ellipsoid of Fig. 2 be combined with two paraboloids with foci at O and O'. Let the first paraboloid be centered around the z-axis, so as to map conformally the plane $z = 0$ onto the wave front S. Similarly, let the second paraboloid map S' onto the plane $z' = 0$. Then, the product of the above three reflections determines a one-to-one correspondence between points U and U' of the two planes $z = 0$ and $z' = 0$ in Fig. 2. This correspondence is everywhere conformal and, therefore, it implies a bilinear relation[2] between the complex coordinates $x + jy$ and $x' + jy'$ of two corresponding points U and U'. It is shown in the following section that the two paraboloids produce the transformations

$$x + jy = 2f_0 u, \qquad x' + jy' = 2f_0' u', \tag{11}$$

f_0 and f_0' being the focal lengths of the two paraboloids. Thus, the desired result, eq. (1), follows at once. As pointed out earlier, eq. (1) requires that one of the two reference systems in Fig. 2 be left-handed.

III. CONSTRUCTION OF A PARABOLOID BY A STEREOGRAPHIC PROJECTION

The mapping between two wave fronts S and S' in Fig. 2 can be represented as a product of two stereographic projections, as shown in Appendix B. In this section, we let O' go to ∞ on the z-axis. Then the ellipsoid degenerates into a paraboloid, S' becomes a plane, and only one stereographic projection is needed.[12]

Let the radius of S be chosen equal to the paraboloid focal length f_0, and let S' be the tangent plane $z = f_0$. Let a correspondence $P \to P'$ between points of S and S' be obtained as shown in Fig. 3, with a stereographic projection from the axial point $z = -f_0$. To show that this is the same correspondence determined by the rays reflected by the paraboloid, consider the reflected ray corresponding to P', and let I be its intersection with the incident ray OP. Since the triangle $PP'I$ is similar to the isosceles triangle NPO, $PI = P'I$ and, therefore,

$$OI = IP' = f_0, \tag{12}$$

which is the equation of a paraboloid.

Notice, if ρ is the radial distance of the reflected ray from the z-axis, then from the triangle $P'VN$ one has

$$\rho = 2f_0 \tan\frac{\theta}{2}. \tag{13}$$

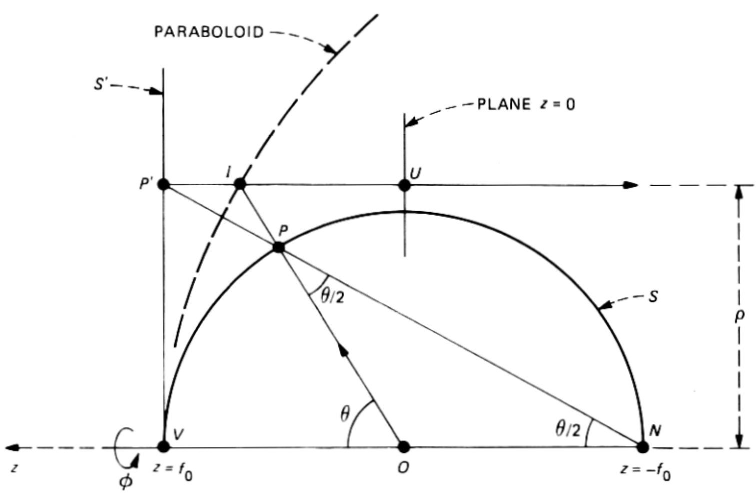

Fig. 3—Construction of a paraboloid by a stereographic projection.

Therefore, the reflected ray determines on the plane $z = 0$ a point U with the complex coordinate given by

$$x + jy = 2f_0 e^{j\phi} \tan \frac{\theta}{2}. \tag{14}$$

which gives eq. (11).

IV. TRANSFORMATION BY AN ELLIPSOID CENTERED AROUND THE Z-AXIS

The usual construction of an ellipse (Fig. 4) involves two fixed points A_1 and A_2 and a variable point A_3 which is varied, keeping the perimeter p of the triangle $A_1A_2A_3$ constant. Then, A_3 describes an ellipse with foci A_1 and A_2, as shown in Fig. 4. A simple relation among the angles of the triangle $A_1A_2A_3$ is given by the following theorem, which is derived in Appendix B with the help of two stereographic projections.

Theorem: Given a triangle $A_1A_2A_3$ with angles α_1, α_2, α_3, and perimeter p, its three sides d_1, d_2, d_3 are given by

$$2d_l = p\left(1 - \tan \frac{\alpha_m}{2} \tan \frac{\alpha_n}{2}\right), \tag{15}$$

where (l, m, n) is any permutation of (1, 2, 3) *and $d_l = A_mA_n$.*

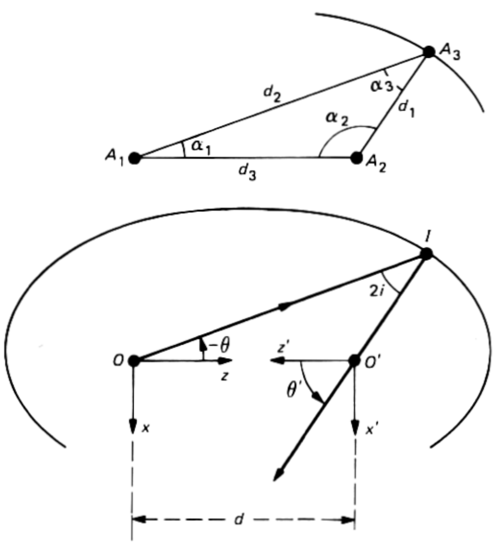

Fig. 4—The angles α_1, α_2, α_3 of a triangle $A_1A_2A_3$ are related by eq. (15), which implies a linear relation between tan $\theta/2$ and tan $\theta'/2$.

By letting A_1 and A_2 coincide with O and O' in Fig. 4, and letting A_3 be the point of incidence of a ray from O, one obtains from eq. (15) for $l = 3$ the well-known relation

$$1 + a_0 \tan\frac{\alpha_1}{2} \tan\frac{\alpha_2}{2} = 0, \tag{16}$$

where a_0 is a constant determined by the distance $d = d_3$ between the two foci and by the path length $t = OIO'$,

$$a_0 = -\frac{t + d}{t - d}. \tag{17}$$

If now the ellipsoid is centered around the z-axis as shown in Fig. 4, one has $\theta = -\alpha_1$. Furthermore, orienting the x', y', z'-axes as in Fig. 4, with the z'-axis opposite the z-axis, we have $\phi' = \phi$ and $\theta' = \pi - \alpha_2$; therefore, eq. (16) gives

$$u' = a_0 u. \tag{18}$$

Thus, for this particular orientation of the reference axes, eq. (1) assumes the normal form of eq. (6). If now arbitrary rotations are applied to the reference axes of Fig. 4, as we have seen in Section II, eq. (18) assumes the form of eq. (1). This implies that the above rotations transform u and u' according to bilinear transformations, whose coefficients are derived next.

V. ROTATIONS AND REFLECTIONS[10]

Consider Fig. 5a showing the x, y, z-axes oriented arbitrarily with respect to the x', y', z'-axes. We wish to determine, for a ray from O, the relationship between its coordinates θ', ϕ' and θ, ϕ with respect to the two coordinate systems. We have seen that the relationship can be written in the form (1), whose coefficients we now express in terms of the Euler angles α, β, γ specifying the orientation of the x', y', z'-axes with respect to the x, y, z-axes. Notice, for the purpose of determining the coefficients of eq. (1), consideration can be restricted to real values of u.

The x', y', z'-axes in Fig. 5a can be obtained from the x, y, z-axes by three successive rotations: a rotation around the z-axis through the Euler angle α, followed by a rotation around the y-axis involving the second Euler angle β and, finally, a rotation around the z-axis by the third Euler angle γ. The first and last rotations are described by the transformations

$$u' = ue^{-j\alpha}, \qquad u' = ue^{-j\gamma}. \tag{19}$$

To determine the second transformation, consider Fig. 5b which illus-

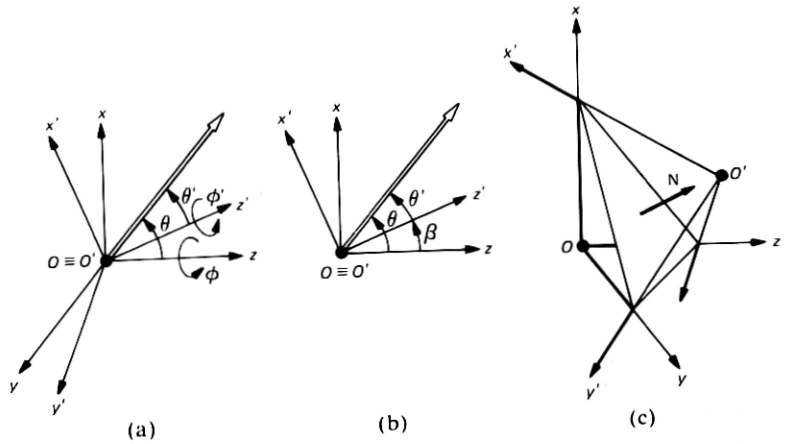

Fig. 5—The x', y', z'-axes are obtained from the x, y, z-axes through (a) an arbitrary rotation, (b) a rotation around the y-axis, and (c) a reflection by a plane.

trates a rotation around the y-axis by the angle β. Then, for a ray in the xz-plane, $\theta' = \theta - \beta$ and, therefore,

$$\tan\frac{\theta'}{2} = \frac{\tan\dfrac{\theta}{2} - \tan\dfrac{\beta}{2}}{1 + \tan\dfrac{\theta}{2}\tan\dfrac{\beta}{2}}, \tag{20}$$

which gives

$$u' = \frac{u + b_0}{1 - b_0 u}, \tag{21}$$

with

$$b_0 = -\tan\frac{\beta}{2}. \tag{22}$$

The product of the above three rotations can now be calculated straightforwardly. Letting

$$b = -\tan\frac{\beta}{2}\, e^{j\alpha}, \tag{23}$$

from eqs. (19), (21), and (22) we obtain

$$u' = e^{-j(\alpha+\gamma)}\frac{b + u}{1 - b^* u}, \tag{24}$$

which represents an arbitrary rotation. In the special case where the axis of rotation is orthogonal to the z-axis, we can verify that

$$\alpha + \gamma = 0 \tag{25}$$

and eq. (24) reduces to

$$u' = \frac{b + u}{1 - b^* u}. \tag{26}$$

Eqs. (24) and (26) can be considered generalizations of the trigonometric identity (20) to complex coordinates. They are directly related to the Cayley-Klein representation[10] of rotations by complex matrices.

5.1 Reflections

We now combine a rotation orthogonal to the z-axis with an inversion of the z-axis and obtain from eq. (26), replacing u with $1/u^*$,

$$u' = \frac{1 + bu^*}{u^* - b^*}, \tag{27}$$

representing a reflection by a plane shown in Fig. 5(c). The x', y', z'-axes in Fig. 5(c) are the reflected images of the x, y, z-axes, and we shall see in a moment that the coefficient b in eq. (27) is given by

$$b = \frac{N_x + jN_y}{N_z}, \tag{28}$$

where N_x, N_y, N_z are the x, y, z-components of a vector $\mathbf{N}$ orthogonal to the reflector.

Eqs. (23) through (28) give the transformation of u when a rotation (or a reflection) is applied to the reference axes. Suppose now the same rotation (or reflection) is applied to a ray with initial coordinate u, so as to obtain a new ray with coordinate u', as in Figs. 6(a) and (b). Then, if both u' and u are measured with respect to the x, y, z-axes, we find* that α, γ, β in eqs. (23) and (24) must be replaced with $-\alpha$, $-\gamma$, $-\beta$, whereas the coefficient b in eq. (27) is still given by eq. (28).

To derive eq. (28), let a reflection be applied to the ray $u = \infty$, as in Fig. 6(c). Then the angle formed by the z-axis and the reflected ray is bisected by $\mathbf{N}$. Thus, since $\mathbf{N}$ is in the plane of incidence, and the angle of $\mathbf{N}$ with respect to the z-axis is $\theta'/2$,

$$u' = \frac{N_x + jN_y}{N_z}, \qquad \text{for} \quad u = \infty.$$

This gives the desired results of eq. (28).

If, instead of a plane, the ray is reflected by an arbitrary surface $z = f(x, y)$, then from eq. (28) one obtains

* To show this, first let u' be measured with respect to the x', y', z'-axes. Then one obtains the identity $u' = u$. Next apply to the x', y', z'-axes the inverse transformation of eq. (24) or (27). The inverse of eq. (24) is a rotation with Euler angles $-\alpha$, $-\gamma$, $-\beta$, whereas the inverse of eq. (27) is a reflection with the same coefficient b of eq. (28).

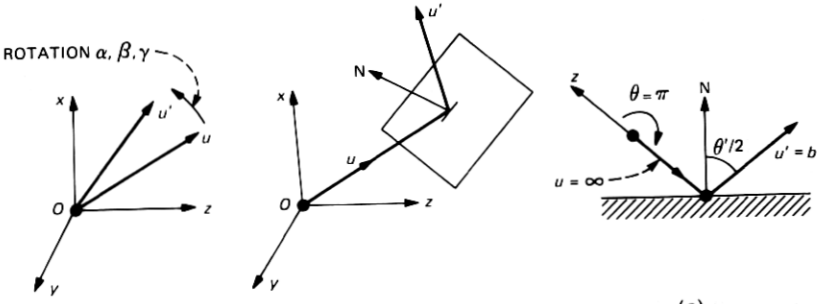

Fig. 6—A ray with initial coordinate u is subjected in (a) to a rotation and in (b) to a reflection by a plane. Notice in (c) $u' = b$ for $u = \infty$.

$$b = -\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) f(x, y), \tag{29}$$

and eq. (27) gives a simple relation between the ray coordinates and the partial derivatives of $f(x, y)$. A similar result[13,14] is obtained if the equation of the reflector is specified in spherical coordinates, as shown in Appendix C.

Notice that eq. (27) applies not only to a ray, but also to its polarization in which case u and u' represent the directions of the incident and reflected polarization with respect to the x, y, z-axes

VI. TRANSFORMATION BY AN ELLIPSOID WHEN THE OUTPUT FRAME IS THE MIRROR IMAGE OF THE INPUT FRAME

In this section, we orient the z-axis in the direction of the principal ray. In a reflector antenna, this is the ray that corresponds to the feed axis. Thus, the principal ray determines the point of maximum illumination over the antenna aperture. To maximize aperture efficiency, the feed is usually oriented so that the principal ray $u = 0$ passes through the center of the aperture. The ray $u = \infty$, which leaves the feed in the direction opposite to the principal ray, will be called the *cardinal ray*.

Consider Fig. 7 which shows an ellipsoid with the principal ray incident at I with angle of incidence i. Let the x', y', z'-axes be the reflected images of the x, y, z-axes with respect to the tangent plane at I. Then, the principal ray after reflection has the direction of the z'-axis, whose complex coordinate $u = \lambda$ with respect to the x, y, z-axes is given by

$$\lambda = \frac{e^{j\psi}}{\tan i},$$

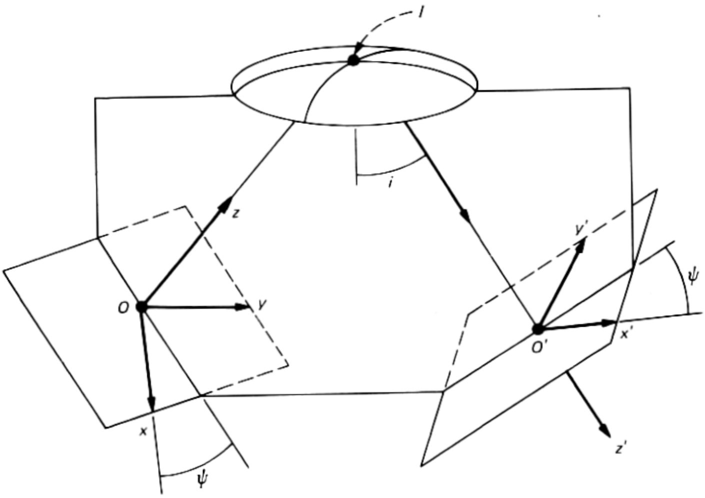

Fig. 7—Reference frames implied by eq. (34). Notice the plane of incidence for the principal ray rotated by ψ with respect to the xz-plane.

ψ being the angle of rotation of the plane of incidence with respect to the x-axis.

Initially, assume $\psi = 0$. Then the x-axis is in the plane of incidence, as shown in Fig. 8, and the same is true for the x'-axis. Thus, both reference systems can be obtained from those of Fig. 4 by suitable rotations around the y-axes, which have the same orientation in both cases. Taking into account that the coefficients of these rotations are real, they transform eq. (18) into

$$u' = a \frac{u}{1 + cu}, \tag{30}$$

where a and c are real coefficients which can be determined as follows. To determine a, consider a ray in the vicinity of the principal ray. Then θ and θ' are small and

$$\theta \ell \simeq -\theta \ell',$$

ℓ and ℓ' being the distances of I from the two foci. It follows that a is equal to the magnification M given by

$$M = -\frac{\ell}{\ell'}. \tag{31}$$

To determine c, let $u = \infty$. We then obtain in Fig. 8 the cardinal ray incident at i' with angle of incidence i'. From the triangle $II'O'$ of Fig. 8,

$$u' = \frac{1}{\tan(i + i')}, \qquad \text{for} \quad u = \infty.$$

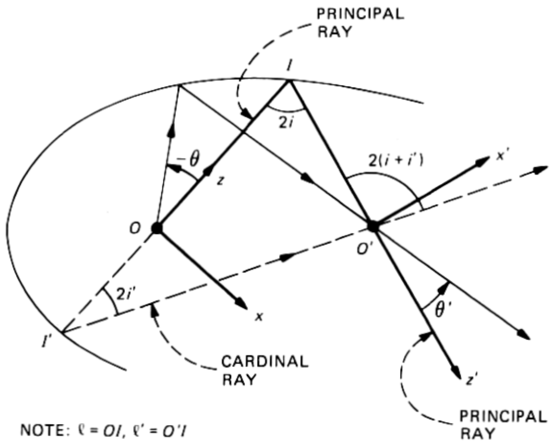

Fig. 8—To obtain eq. (33), each reference system must be oriented so that the z-axis is in the direction of the principal ray, and the x-axis is in the plane of incidence.

Furthermore, applying eq. (15) to the triangle $II'O'$ with perimeter $p = 2t = 2(\ell + \ell')$,

$$(\ell + \ell')\tan i = \ell \tan(i + i'), \tag{32}$$

which gives the desired result,

$$c = (M - 1)\tan i.$$

Finally,

$$u' = M \frac{u}{1 + (M - 1)u \tan i}, \tag{33}$$

which assumes the x-axis is in the plane of incidence, so that $\psi = 0$.

Now consider the general case $\psi \neq 0$. Then both reference systems in Fig. 8 must be rotated around the z-axes by $-\psi$, and from eq. (33) we obtain

$$u' = M \frac{u}{1 + u(M - 1)e^{-j\psi}\tan i}, \tag{34}$$

which applies in general when the plane of incidence is rotated by an arbitrary angle ψ with respect to the xz-plane.

Using this simple relation, we can now determine straightforwardly how the nonzero angle of incidence i in Fig. 7 affects the amplitude pattern of the reflected wave, its polarization, its symmetry, and the aberrations arising when the point source is slightly displaced from O. For $i = 0$, eq. (34) reduces to eq. (18). In this case the transformation has circular symmetry, since it is unaffected if identical rotations are applied to the reference systems around the z-axes. For $i \neq 0$, on the other hand, eq. (34) lacks this symmetry. We now show that, by properly combining several asymmetric transformations of the type (34), it is always possible to obtain the symmetric transformations (18). This was first shown in Refs. (15) and (16) for two reflectors with O' at ∞ and, in Ref. (17) under more general conditions.

VII. TRANSFORMATION BY A SEQUENCE OF ELLIPSOIDS

Replace the ellipsoid of Fig. 7 with a sequence of ellipsoids, with foci $O_0, O_1, \cdots, O_N$ as shown in Fig. 9. Let the $(s+1)$th reference frame be the mirror image of the sth frame as in Section VI. Let M_s, i_s, ψ_s be the values of M, i, ψ for the sth ellipsoid. Then, the product of the N transformations of Fig. 9 gives eq. (34) with

$$M = M_1 \cdots M_N \tag{35}$$

and

$$(M-1)e^{-j\psi}\tan i = (M_1 - 1)e^{-j\psi_1}\tan i_1 + (M_2 - 1)M_1 e^{-j\psi_2}\tan i_2 + \cdots \tag{36}$$

We have thus shown that eq. (34), derived in Section VI for the ellipsoid of Fig. 7, applies also to a sequence of N ellipsoids. It also

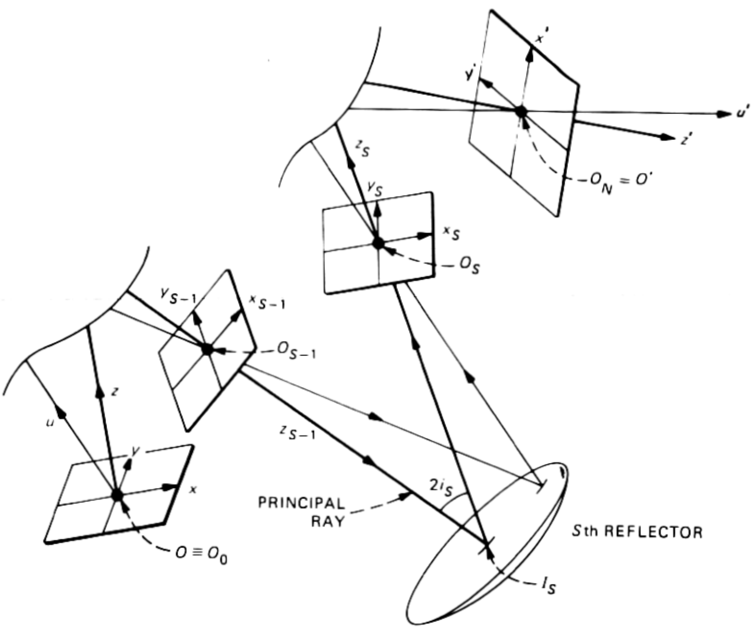

Fig. 9—An input ray with coordinated u is transformed by a sequence of N reflectors into an output ray with coordinate u' given by eq. (34). The principal ray for $u = 0$ is reflected by the s^{th} reflector at I_s with angle of incidence i_s.

applies to a hyperboloid, in which case $M < 0$ since then one of the two foci O and O' in Fig. 7 is behind the reflector and therefore either ℓ or ℓ' is negative. For

$$M \to \infty,$$

the ellipsoid of Fig. 7 degenerates into a paraboloid, shown in Fig. 1 for $\psi = 0$. Then, from eq. (34), letting

$$M \to 0, \qquad u' \to M\frac{w}{2\ell} \tag{37}$$

we obtain

$$w = 2f\frac{u}{1 - u \tan i\, e^{-j\psi}}, \tag{38}$$

where $f = \ell$, and

$$w = x' + jy' \tag{39}$$

is the complex coordinate intercepted in Fig. 1 by the reflected ray on the plane $z' = 0$.

Equation (38) also applies to the arrangement of Fig. 9, with the last ellipsoid replaced by a paraboloid, in which case ψ and $\tan i$ are given by eq. (36) with

$$M = M_N = 0. \tag{40}$$

Then f in eq. (38) is the *equivalent focal length* given by

$$f = M_e \ell_N, \tag{41}$$

where M_e is the magnification determined by the first $N - 1$ reflectors,

$$M_e = M_1 \cdots M_{N-1}, \tag{42}$$

and ℓ_N is the focal length of the last paraboloid.

As pointed out in the introduction, it is desirable in general that

$$\tan i = 0, \tag{43}$$

because then the transformation has circular symmetry with respect to the principal ray. From eq. (36) for $N = 2$, this requires

$$\psi_1 = \psi_2 \tag{44}$$

and

$$\tan i_1(M_1 - 1) + \tan i_2(M_2 - 1)M_1 = 0. \tag{45}$$

The first condition demands that the two planes of incidence (for the principal ray) coincide, in which case the two reflectors and the feed have a common plane of symmetry. In general, for arbitrary N, one finds that it is always possible to satisfy condition (43) by properly choosing one of the planes of incidence and one of the angles i_s for any arbitrary choice of the remaining i_s. The correct choice for i_s is obtained straightforwardly using eq. (36). In some cases, it may not be possible to satisfy exactly the requirement (43). For instance, the N reflectors may have to fit inside a satellite and, because of the limited available space, it may be convenient to choose $i \neq 0$. Then, the resulting aberrations and distortion of the polarization and amplitude illumination over the aperture are determined straightforwardly using eq. (34), as pointed out in Ref. 19.

VIII. GEOMETRICAL DERIVATION OF TAN *I* WHEN THE LAST REFLECTOR IS A PARABOLOID

Assume the final reflector is a paraboloid, and let the input point source be a corrugated feed,[20] or a feed with similar radiation characteristics.[21,22] Then, the spherical wave radiated by the feed has an axis of circular symmetry, and its polarization lines on a wave front S are given by a set of tangent circles as shown in Fig. 10. The contact point D for these circles is one of the two intersections of the feed axis with the wave front S. The other intersection C is the point of maximum illumination. Thus, C and D are determined by the principal ray ($u = 0$) and the cardinal ray ($u = \infty$), respectively. It is now recalled that the bilinear transformation (1) transforms circles into circles. This means that the polarization lines of an output wave front S' are also a set of tangent circles. Their contact point D' is determined by the

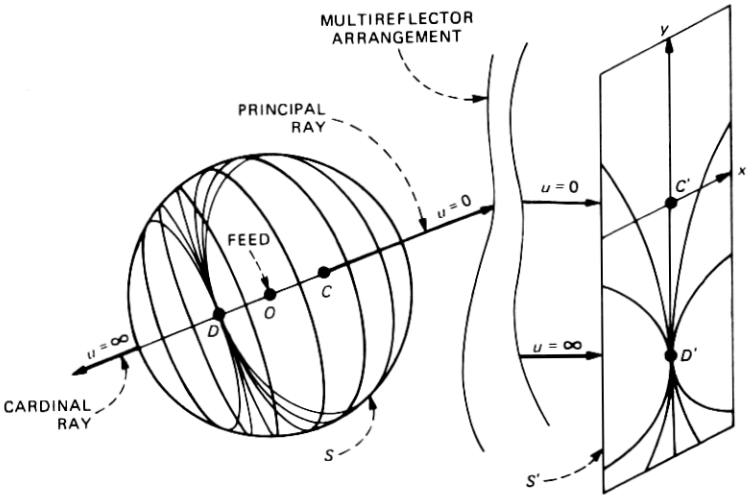

Fig. 10—The polarization lines produced by a corrugated feed are tangent circles with contact point determined by the cardinal ray.

ray $u = \infty$, and the point of maximum illumination C' is determined by the ray $u = 0$. From eq. (8) for $u \to \infty$, the distance of C' from D' is given by

$$CD' = \frac{2f}{\tan i} \tag{46}$$

and, therefore, for $\tan i \to 0$

$$D' \to \infty.$$

Then the circles degenerate into parallel lines, and the output wave front S' becomes everywhere polarized in one direction.[23]

The above considerations suggest a simple procedure for determining the feed axis orientation that corresponds to $\tan i = 0$. The feed must be oriented so that $D' \to \infty$. This means that the cardinal ray must be reflected at ∞ by the last reflector (a paraboloid). Thus, the cardinal ray after the first $N - 1$ reflections must pass through the paraboloid focus O_{N-1} with direction opposite to $O_{N-1}V$, where V is the paraboloid vertex. Therefore, one must orient the feed axis so that the cardinal ray ($u = \infty$) produces after $N - 1$ reflections the ray VO_{N-1}. Since the final direction of this ray is specified, the initial direction can be determined by retracing the ray in the reverse direction starting from O_{N-1}. Then, after $(N - 1)$ reflections of the ray $O_{N-1}V$, one obtains through 0 the direction of OC characterized by $\tan i = 0$. This geometrical derivation is illustrated in Ref. 17.

IX. AMPLITUDE AND POLARIZATION OF THE OUTPUT WAVE

Consider two corresponding points P and P' on the two wave fronts S and S' of Fig. 2. Let the electric field at P be given by

$$\mathbf{E} = A\mathbf{e}\frac{e^{-jkr}}{r}, \tag{47}$$

where r is the distance from the focus O and $\mathbf{e}$ is a unit vector specifying the polarization of $\mathbf{E}$. Similarly, for the field at P'

$$\mathbf{E} = -A'\mathbf{e}'\frac{e^{-jk(r'+t)}}{r'}, \tag{48}$$

where $t = \ell + \ell'$. Let

$$\mathbf{e} = \cos\phi_e \mathbf{i}_1 + \sin\phi_e \mathbf{i}_2, \tag{49}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$ are unit vectors in the θ, ϕ-directions and ϕ_e is the angle of rotation of $\mathbf{e}$ with respect to $\mathbf{i}_1$. In this section, we show that the corresponding angle of rotation ϕ_e' for $\mathbf{e}'$ with respect to $\mathbf{i}_1'$ is simply given by

$$\phi_e' - \phi_e = \phi' - \phi, \tag{50}$$

where it is recalled that ϕ' and ϕ are the arguments of the coordinates u' and u, respectively. For the amplitude A' we show that

$$A' = \frac{1}{m} A, \tag{51}$$

where the magnification m is given by

$$m = \frac{1}{M} \frac{u'u'^*(1 + uu^*)}{uu^*(1 + u'u'^*)}. \tag{52}$$

These relations apply in general to an arbitrary sequence of ellipsoids, hyperboloids, and paraboloids arranged as in Fig. 9. If the first focus O is at infinity, then S is a plane and the spherical coordinates θ, ϕ must be replaced with polar coordinates ρ, ϕ. Then, $\mathbf{i}_1$ is a unit vector in the ρ-direction. Similar considerations apply if O' is at ∞.

To derive eqs. (50) and (52), it is convenient to combine the ellipsoid of Fig. 2 with two paraboloids, as in Section II. Then, one paraboloid maps conformally the sphere S onto the plane $z = 0$ and the other paraboloid the sphere S' onto the plane $z' = 0$. Let A_0, $\mathbf{e}_0$ and A_0', $\mathbf{e}_0'$ be the values of A, $\mathbf{e}$ produced on the two planes, and assume the two mappings are characterized by the transformations

$$u = x + jy, \qquad u' = x' + jy',$$

which imply $f_0 = f_0' = 1/2$ in eq. (11). These transformations do not affect the polarization angles ϕ_e and ϕ_e', while the amplitude A is transformed according to the well-known relation

$$A_0 = \frac{A}{1 + \tan^2 \dfrac{\theta}{2}} = \frac{A}{1 + uu^*}, \tag{53}$$

and similarly for A_0'. According to geometric optics, conservation of power requires

$$|A_0|^2 d\sigma_0 = |A_0'|^2 d\sigma_0', \tag{54}$$

where $d\sigma_0$ and $d\sigma_0'$ are the areas of two corresponding elements of the two planes. Since the mapping between the two planes is conformal,

$$\frac{d\sigma_0}{d\sigma_0'} = \left| \frac{du'}{du} \right|^2, \tag{55}$$

and using eqs. (34), (53), and (54), one obtains the desired result, eq. (52).

Next, we derive eq. (50). Consider two corresponding points Q and Q' of the two planes. Let the polarization line through Q be a straight line through the origin, as in Fig. 11. Then $\phi_e = 0$, and the corresponding polarization line through Q' is a circle. The circle must pass through the origin O', and also through the point D' of coordinate

$$u'_\infty = \frac{M}{M-1}\frac{1}{\tan i}e^{j\psi}, \tag{56}$$

which corresponds to the point at ∞ of the u-plane, as one can verify from eq. (34) letting $u \to \infty$. Now the angle made in Fig. 11 by the chord $D'Q'$ with the tangent $\mathbf{e}'_0$ is equal to the angle β subtended by the chord at O'. As a consequence, one can verify that the angle ϕ'_e in Fig. 11 is equal to the angle γ between $D'Q'$ and $D'O'$. Thus,

$$\gamma = \phi'_e - \phi_e = \angle\left[\frac{u' - u'_\infty}{-u'_\infty}\right] = \angle\left[1 - \tan i\,\frac{M-1}{M}u'e^{-j\psi}\right], \tag{57}$$

and one can verify that this agrees with eq. (50). Using eqs. (50) through (52), one can now derive straightforwardly the amplitude and polarization of the output wave in Fig. 9.

X. CONCLUSIONS

With simple geometric considerations, we have derived the coefficients of the transformation (1). Once the parameters i_s, M_s, ψ_s that specify the path of the principal ray are known, the coefficients can be derived straightforwardly using eqs. (34) and (36). For a corrugated feed, it has been shown in Section VIII that the circles describing the polarization of an output wave front can be determined straightfor-

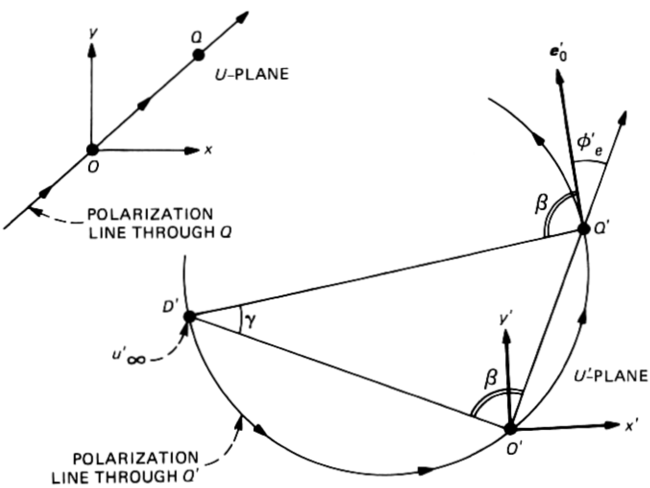

Fig. 11—Derivation of the polarization in the u'-plane when the u-plane is polarized in the ρ-direction.

wardly by tracing the cardinal ray. In Section V, it has been shown how the transformation (35) is affected by a rotation of the feed axis. The results will be useful to the design of reflector antennas as pointed out in Ref. 19. They also provide a simple interpretation for previous results of Refs. 13 and 19, as pointed out in Appendix C.

APPENDIX A

Consider Fig. 2. We wish to show that if L is everywhere tangent to the magnetic field, then this is true also for L'. Suppose initially that S and S' are both centered at O. Then, the mapping determined between S and S' by a ray from O is just a similarity, with magnification determined by the radii of S and S'. This means that each line element $\delta L'$ of L' is parallel* to the corresponding line element δL of L. Thus, if both S and S' are centered at O, or both at O', then the two curves L and L' certainly satisfy (10).

Next, let S and S' be centered at O and O', respectively, and let I be the point of incidence for the ray corresponding to δL. Then, the orientation of the corresponding line element $\delta L'$ is not affected if the ellipsoid in Fig. 2 is replaced by the tangent plane at I. Thus, we conclude that property (10) is true in general, even if S' and S are centered at different foci.

Notice (10) implies that the mapping between any two wave fronts S and S' preserves angles and, therefore, it is conformal.

APPENDIX B

We show that the mapping in Fig. 2 between the two wave fronts S and S' can be represented as a product of two stereographic projections. A variety of different representations can be obtained, depending on the radii r and r' of S and S'. For simplicity, here we choose the two radii so that the two spheres S and S' touch each other, as shown in Fig. 12. The contact point V is on the axis OO' of the ellipsoid. Let N and N' be the other intersections of the two spheres with the axis. Let P_1 be an arbitrary point of the tangent plane at V, and let two corresponding rays OP and $O'P'$ be obtained as shown in Fig. 12, with two stereographic projections from N and N', respectively. We now show that the intersection I of the two rays satisfies the condition

$$OI + IO' = r + r', \tag{58}$$

which is the equation of an ellipsoid. Let θ and θ' be the angles VOP and $VO'P'$, respectively. Then the isosceles triangle ONP has two of

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* This property, is true *only if* S and S' are spherical wave fronts or if L is a geodesic line of S.

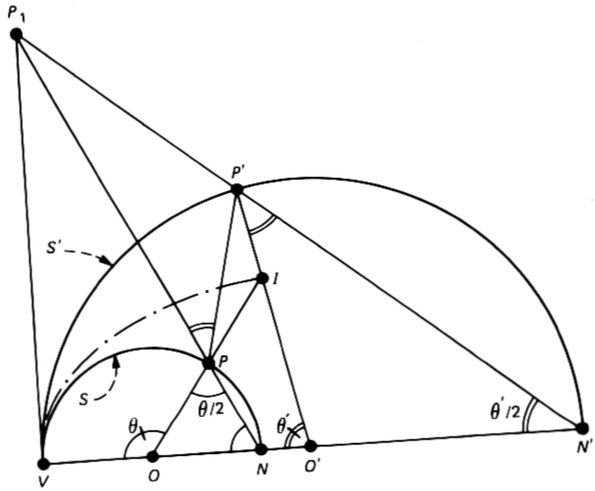

Fig. 12—A point I of an ellipse with foci O and O' is obtained with two stereographic projections from N and N'.

its angles equal to $\theta/2$, and similarly the triangle $O'N'P'$ has two angles equal to $\theta'/2$. Furthermore, since VP_1 is tangent to both spheres,

$$NP_1 \times P_1P = N'P_1 \times P_1P',$$

since both products must equal VP_1^2. Thus, the triangles PP_1P' and NP_1N' are similar and, therefore, $P_1PP' = \theta'/2$, $PP'P_1 = 180° - \theta/2$. The angles $PP'I$ and $P'PI$ can now be determined, and one finds they are both equal to $(\theta + \theta')/2$. Thus, $PI = P'I$, which gives eq. (58).

From the right triangles P_1VN and P_1VN', since they have one side in common,

$$r \tan \frac{\theta}{2} = r' \tan \frac{\theta'}{2}, \tag{59}$$

which gives eq. (16), and this implies the theorem of Section IV.

APPENDIX C

We now point out a simple connection between eqs. (28) and (29) and previous results by Brickell and Westcott.[13,14] Both u and u' will be measured with respect to the same reference frame, the x, y, z-axes.

Let an arbitrary reflector be illuminated by a spherical wave from the origin O of the x, y, z-axes, and let the reflecting surface be given in spherical coordinates by

$$\rho = \rho(u, u^*),$$

where ρ is the distance from the origin O, and $\rho(u, u^*)$ is an analytic function† of u, u^*. Consider a particular incident ray with coordinate $u = \lambda$, and let P be its point of incidence. To determine its reflected coordinate u' with respect to the x, y, z-coordinates, it is convenient to introduce temporarily a second reference frame with the z-axis through the point of incidence. Then, using the subscript $(\ )_0$ for the second frame, and applying eqs. (27) and (29) to the ray through P,

$$u'_0 = \rho\left(\frac{\partial\rho}{\partial u_0}\right)^{-1}, \qquad \text{for} \quad u_0 = 0. \tag{60}$$

Next, we apply a suitable rotation to the x_0, y_0, z_0-axes, so as to transform u'_0, u_0 into u', u,

$$u' = \frac{u'_0 + \lambda}{1 - u'_0\lambda^*}, \qquad u = \frac{u_0 + \lambda}{1 - u_0\lambda^*}.$$

From the second expression for $u_0 = 0$,

$$\frac{\partial u}{\partial u_0} = (1 + uu^*), \qquad \frac{\partial u^*}{\partial u_0} = 0,$$

and, therefore,

$$\frac{\partial}{\partial u_0} = \frac{\partial u}{\partial u_0}\frac{\partial}{\partial u} + \frac{\partial u^*}{\partial u_0}\frac{\partial}{\partial u^*} = (1 + uu^*)\frac{\partial}{\partial u},$$

for $u_0 = 0$. From these relations, taking into account that $u = \lambda$ for $u_0 = 0$, one obtains the final result

$$u' = \frac{\rho + (1 + uu^*)u\dfrac{\partial\rho}{\partial u}}{(1 + uu^*)\dfrac{\partial\rho}{\partial u} - u^*\rho}, \tag{61}$$

valid for any u. This gives eq. (16) of Ref. 13. We have thus shown that this basic result can be considered a direct consequence of eqs. (27) and (29).

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† It is convenient to express ρ in terms of both u and u^*, since the real quantity ρ is not an analytic function of u. This is best understood by expanding ρ in a power series of u. Then, since ρ is real and u is complex, both powers of u and u^* must be considered.

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