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On Newton-Direction Algorithms and Diffeomorphisms*

By I. W. SANDBERG

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This paper reports on results that complement those in an earlier paper by this writer which gives a constructive proof of the existence of an algorithm that, for each right-hand side a, produces a sequence which converges globally and superlinearly to a solution x of f(x) = a whenever f is a C^1-diffeomorphism (i.e., is a continuously-differentiable invertible map with continuously-differentiable inverse) of a Banach space B onto itself and either $B = R^n$ or f satisfies certain other conditions that are often met in applications. Here we consider the case in which f′ is Lipschitz on each bounded subset of B. We give results which, while along the lines of those obtained earlier, concern a fundamentally different Newton-direction algorithm which does not appear to have been introduced previously, and which has the advantage that its implementation does not require the use of certain search procedures.

I. INTRODUCTION

Let f be a function from U into B, where B is a Banach space with norm $|\cdot|$, and U is a nonempty open subset of B. We say that f is *differentiable* on a set $S \subset U$ if f has a Frechet derivative $f'(s)$ at each point s of S.† (If, for example, $B = R^n$ with the usual Euclidean norm, then f is differentiable on U if it is continuously differentiable on U in the usual sense.) By f a *C^1-diffeomorphism*, we mean that f is a homeomorphism of U onto B, and f' and $(f^{-1})'$ exist and are continuous on U and B, respectively. (We emphasize that here *continuity* refers to the dependence of the derivatives on the points at which they are

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† In other words, f is *differentiable* on $S \subset U$ if for each $s \in S$, there is a bounded linear map $f'(s): B \to B$ such that $f(s + h) = f(s) + f'(s)h + o(|h|)$ as $|h| \to 0$.

evaluated, not to their boundedness as operators, which is assured by definition*) C^1-diffeomorphisms frequently arise in applications.

The purpose of this paper is to report on results that complement those in Ref. 1 where a constructive proof is given of the existence of a Newton-direction algorithm that, for each a $\in B$, generates a sequence in U which converges globally and superlinearly to a solution x of $f(x) = a$ whenever f is a C^1-diffeomorphism of U onto B and either $B = R^n$ or f satisfies certain other conditions that are frequently met in applications. (For the case of an important class of *monotone* diffeomorphisms f in a Hilbert space H, the "other conditions" reduce to simply the requirement that f' be uniformly continuous on closed bounded subsets of H. A specific example in which H is infinite dimensional is given in Ref. 1.)

The algorithm described in Ref. 1 typically involves the recursive determination of positive scalars $\gamma_0, \gamma_1, \cdots$ (which determine the successive steplengths) such that a certain ratio $R_k(\gamma_k)$ (which depends on the kth iterate x^k) lies between prescribed bounds for all $k = 0, 1, 2, \cdots$. While it is proved that the γ_k can be chosen as required, and that $\gamma_k = 1$ for all sufficiently large k, the actual determination of the γ_k in a specific case would ordinarily require the use of a one-dimensional search procedure for a finite (and possibly large) number of values of k.

In this paper we address the case in which $U = B$ and f' is *Lipschitz* on bounded subsets of B (i.e., is such that for each bounded subset S of B there is a constant Λ such that $|f'(u) - f'(v)| \leq \Lambda |u - v|$ for all u and v in S). We give results which, while along the lines of those in Ref. 1, concern a fundamentally different Newton-direction algorithm that does not appear to have been introduced earlier, and which does not require the use of search procedures to solve subproblems of the type outlined above.

Our results are presented in Section II. As a consequence of the Lipschitz hypothesis, proofs are comparatively simple and we are able to establish quadratic (rather than superlinear) convergence. (Recall that a sequence $x^1, x^2, \cdots$ in B *converges quadratically* to an element x of B if the sequence converges to x and there is a constant c such that $|x^{k+1} - x| \leq c|x^k - x|^2$ for all k.†)

General relationships between diffeomorphisms and computation of the type described in Ref. 1 and in Section II do not appear to have

* And of course, this continuity is with respect to the usual induced norm of a bounded linear map of B into B.

† Quadratic convergence results follow easily from those in Ref. 1 under the Lipschitz hypothesis used here. (In this connection, see the last part of the proof of Lemma 1 in Section II.)

been reported on earlier by other writers. On the other hand, as in Ref. 1 our approach involves the minimization of a functional, and therefore in a general sense there is a vast related literature. [See, for example, Ref. 2 and note (p. 190) that the least-squares Newton-direction methods described there require, in particular, the existence of *second* derivatives of f (our notation).] Additional background material can be found in Ref. 1.

II. PROCESSES N_0 and N_1

Throughout this section we use the terms *Lipschitz* and *converges quadratically* in the way indicated in Section I, we denote the usual induced norm of a linear map A of B into B by $|A|$, and we take $U = B$.

With f differentiable on B, but not necessarily a C^1-diffeomorphism, and with x^0 and a any two elements of B, consider the following process, in which s_k denotes $|f(x^k) - a|$ whenever $x^k \in B$ is defined.

Process N_1: Choose $\rho \in [\frac{1}{2}, 1)$ and $\lambda > 0$. Do the following for $k = 0, 1, \cdots$

If $f(x^k) = a$, set $x^{k+1} = x^k$.

If $f(x^k) \neq a$, determine $\phi_k \in B$ such that

$f'(x^k)\phi_k = a - f(x^k)$. Then

1. Let $\gamma_k = (\lambda s_k)^{-1}$ if $s_k > 2\rho\lambda^{-1}$

$= 1$ if $s_k \leq 2\rho\lambda^{-1}$.

2. Let $y^{k+1} = x^k + \gamma_k\phi_k$.

3. Set $x^{k+1} = y^{k+1}$ if either $s_k > 2\rho\lambda^{-1}$ and $|f(y^{k+1}) - a| \leq [1 - (2\lambda s_k)^{-1}]s_k$, or $s_k \leq 2\rho\lambda^{-1}$ and $|f(y^{k+1}) - a| \leq \frac{1}{2}\lambda s_k^2$. If neither pair of conditions is met, replace λ by 2λ in Step 1 and the sentence preceding this sentence, and return to Step 1.

Our main result is the following.

Theorem 1: Suppose that f is a C^1-diffeomorphism of B onto B. Let f' be Lipschitz on bounded subsets of B, and let $|(f^{-1})'|$ be bounded on bounded subsets of B. Then for each a and each x^0, Process N_1 can be carried out, and x^1, x^2, $\cdots$ converges quadratically to the unique solution x of $f(x) = a$.

2.1 Proof of Theorem 1

Let a and x^0 be given.

We first prove two lemmas which concern cases in which f need not be a C^1-diffeomorphism. Let $L = \{v \in B: |f(v) - a| \leq |f(x^0) - a|\}$, and let $\bar{L}$ denote $\{w + \alpha f'(w)^{-1}[a - f(w)]: w \in L, \alpha \in [0, 1]\}$ when $f'(\cdot)^{-1}$ exists on L. (Assuming that $\bar{L}$ is defined, notice that it is bounded if L is bounded and $|f'(\cdot)^{-1}|$ is bounded on L. This observation is used later in the proof of Theorem 1 and in connection with Lemmas 1 and 2, below.) With η a positive constant, consider the following process.

Process N_0: Choose $\rho \in [½, 1)$. Do the following for $k = 0, 1, \cdots$.

If $f(x^k) = a$, set $x^{k+1} = x^k$.

If $f(x^k) \neq a$, determine $\phi_k \in B$ such that

$f'(x^k)\phi_k = a - f(x^k)$. Then let

$$\gamma_k = (\eta s_k)^{-1} \quad \text{if} \quad s_k > 2\rho\eta^{-1}$$
$$= 1 \quad \text{if} \quad s_k \leq 2\rho\eta^{-1}.$$

Set $x^{k+1} = x^k + \gamma_k \phi_k$.

Lemma 1: Assume that L is bounded and that $f'(\cdot)$ and $f'(\cdot)^{-1}$ exist on L with $|f'(w)^{-1}| \leq K$ for $w \in L$ and some constant K. Assume also that $f'(\cdot)$ exists and is Lipschitz, with Lipschitz constant Λ, on $\bar{L}$. Then for $\eta \geq \Lambda K^2$,

(a) Process N_0 can be carried out, and for each k such that $s_k \neq 0$ we have

$$s_{k+1} \leq [1 - (2\eta s_k)^{-1}]s_k \quad \text{if} \quad s_k > 2\rho\eta^{-1}, \tag{1}$$

$$s_{k+1} \leq ½\, \eta s_k^2 \leq \rho s_k \quad \text{if} \quad s_k \leq 2\rho\eta^{-1}. \tag{2}$$

(b) $s_k \to 0$ as $k \to \infty$.

(c) If there is an $x \in B$ such that $f(x) = a$ and $x^k \to x$ as $k \to \infty$, then $\{x^k\}$ converges quadratically.

2.1.1 Proof of Lemma 1

We will use the following proposition.

Proposition 1: Suppose that the hypotheses of Lemma 1 are met. If $x^k \in L$, and $\gamma \in [0, 1]$, and ϕ_k denotes $f'(x^k)^{-1}[a - f(x^k)]$, then, for $\eta \geq \Lambda K^2$, we have $|f(x^k + \gamma\phi_k) - a| \leq (1 - \gamma)|f(x^k) - a| + ½\eta\gamma^2|f(x^k) - a|^2$.

Proof: We have

$$|f(x^k + \gamma\phi_k) - a| = |f(x^k) - a + f'(x^k)\gamma\phi_k + \delta|$$

in which

$$\delta = f(x^k + \gamma\phi_k) - f(x^k) - f'(x^k)\gamma\phi_k.$$

Thus

$$|f(x^k + \gamma\phi_k) - a| \le (1 - \gamma)|f(x^k) - a| + |\delta|,$$

and

$$\delta = \int_0^1 [f'(x^k + \beta\gamma\phi_k) - f'(x^k)]\, d\beta \cdot \gamma\phi_k.$$

Since $|\delta| \le \frac{1}{2}\Lambda\gamma^2 K^2 |f(x^k) - a|^2$, we have proved the proposition.*

Assume now that $\eta \ge \Lambda K^2$, and that the hypotheses of Lemma 1 are met.

Let k be such that either $k = 0$, or Process N_0 can be used to generate $x^1, \cdots, x^k$ with $x^j \in L$ for $j = 1, 2, \cdots, k$. Suppose that $s_k \ne 0$. Since $x^k \in L$, ϕ_k can be determined. Since $\rho \in [\frac{1}{2}, 1)$, when $s_k > 2\rho\eta^{-1}$ we have $(\eta s_k)^{-1} < 1$. Thus, by the proposition, (1) holds. On the other hand, obviously $\frac{1}{2}\eta s_k \le \rho$ when $s_k \le 2\rho\eta^{-1}$, and thus, by the proposition, (2) is met. This shows that x^{k+1} can be determined, that it satisfies (1) and (2), and that $x^{k+1} \in L$, which proves Part (a).

Part (b) is a direct consequence of Part (a), because, by Part (a), if s_k does not approach zero as $k \to \infty$ we must have $s_k > 2\rho\eta^{-1}$ for all k in which case $[1 - (2\eta s_0)^{-1}] \in (0, 1)$ and $s_k \le [1 - (2\eta s_0)^{-1}]^k s_0$ for $k \ge 1$, which is a contradiction.

Assume now that B contains an x such that $f(x) = a$ and $x^k \to x$ as $k \to \infty$.† Since $x^k \in L$ for all k, and L is closed, $x \in L$. Let J denote $f'(x)$. Since J is an invertible bounded linear map of B into itself, there are positive constants β_1 and β_2 such that $\beta_1|u| \le |Ju| \le \beta_2|u|$ for $u \in B$. For each k, we have

$$|f(x^k) - a| = |f(x) - a + J(x^k - x) + \delta_k|$$

in which $|\delta_k|(|x^k - x|)^{-1} \to 0$ as $k \to \infty$.

Notice that for some m,

$$|J(x^k - x)| \ge 2|\delta_k| \qquad \text{for} \qquad k \ge m.$$

Thus for $k \ge m$,

$$|f(x^k) - a| \ge |J(x^k - x)| - |\delta_k| \ge \tfrac{1}{2}|J(x^k - x)| \ge \tfrac{1}{2}\beta_1|x^k - x|,$$

and, on the other hand,

$$|f(x^k) - a| \le |J(x^k - x)| + |\delta_k| \le \tfrac{3}{2}|J(x^k - x)| \le \tfrac{3}{2}\beta_2|x^k - x|.$$

We have $s_{k+1} \le \frac{1}{2}\eta s_k^2$ for $k \ge M$ for some $M \ge m$.

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* With regard to the origin of the formula for γ_k in Process N_0, notice that the right side of the main inequality of the proposition is minimized with respect to γ at $\gamma = (\eta s_k)^{-1}$.

† The existence of such an x follows from our hypotheses, but this fact is not needed for our purposes.

Therefore,

$$|x^{k+1} - x| \leq 9/4\, \eta\beta_1^{-1}\beta_2^2 |x^k - x|^2, \qquad k \geq M,$$

which completes the proof of the lemma.*,†

Lemma 2: Suppose that L is bounded, and that $f'(\cdot)$ and $f'(\cdot)^{-1}$ exist on L with $|f'(\cdot)^{-1}|$ bounded on L. Suppose also that $f'(\cdot)$ exists and is Lipschitz on $\bar{L}$. Then Process N_1 can be carried out, we have $s_k \to 0$ as $k \to \infty$, and if there is an $x \in B$ such that $f(x) = a$ and $x^k \to x$ as $k \to \infty$, then $x^1, x^2, \cdots$ converges quadratically.

2.1.2 Proof of Lemma 2

Consider Process N_1. By Lemma 1, there is a constant λ_0 that depends only on f, a, and x^0 such that if λ in Step 1 and the first sentence of Step 3 satisfies $\lambda \geq \lambda_0$, and if either $k = 0$ and $s_0 \neq 0$, or $k > 0$ and Process N_1 can be used to determine x^k with $s_k \neq 0$ and $s_k \leq s_0$, then Step 3 can be carried out on the first pass. Notice that whenever x^{k+1} is set equal to y^{k+1} in Step 3, we have $s_{k+1} < s_k$.

Since for any $\lambda > 0$ there is a nonnegative integer p such that $2^r\lambda \geq \lambda_0$, it follows that Process N_1 can be carried out, and that for some nonnegative integers q and r, we have $s^{k+1} \leq \sigma_k s^k$ for $k \geq q$, where $\sigma_k = [1 - (2^{r+1}\lambda s_q)^{-1}]$ when $s^k > 2\rho(2^r\lambda)^{-1}$ and $\sigma_k = \rho$ otherwise. Since $\sigma_k < 1$ for $k \geq q$, it is clear that $s_k \to 0$ as $k \to \infty$, and therefore that $s_{k+1} \leq \frac{1}{2}\, 2^r\lambda s_k^2$ for $k \geq M$ for some M. Thus, by the proof of Part (c) of Lemma 1, our proof of Lemma 2 is complete.

Now let the hypotheses of Theorem 1 be met. The proof of Theorem 3 of Ref. 1 shows that L is bounded, that $f'(\cdot)^{-1}$ exists on B, and that $|f'(\cdot)^{-1}|$ is bounded on L. Since f is a homeomorphism of B onto B, $s_k \to 0$ as $k \to \infty$ implies that $x^k \to x$ as $k \to \infty$, where x satisfies $f(x) = a$. By Lemma 2, this completes the proof of Theorem 1.

2.2 Monotone diffeomorphisms in Hilbert space

Let $\psi:[0, \infty) \to [0, \infty)$ be continuous, strictly increasing, and such that $\psi(0) = 0$, $\psi(\alpha) \to \infty$ as $\alpha \to \infty$, and $\alpha^{-1}\psi(\alpha) \geq c$ for $\alpha \in (0, \bar{\alpha})$ for some positive constants c and $\bar{\alpha}$. Notice that, for example, $\psi(\alpha) = \alpha$ meets these conditions.

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* The fact that $\{x^k\}$ converges quadratically follows from a direct extension of a known result (see Ref. 3, p. 312) since either $s_k = 0$ for some k, or there is an M such that $\gamma_k = 1$ for $k \geq M$. The short proof given above is included for the sake of completeness.

† D. J. Rose has informed this writer that in recent independent joint work with R. Bank,[4] done subsequent to the appearance of preprints of Ref. 1, a corresponding result, as well as a result corresponding to Theorem 1, was obtained for a process in which $\gamma_k = (1 + \eta_k s_k)^{-1}$, where the η_k satisfy certain inequalities. They study a case in which an approximation M_k to $f'(x^k)$ can be used in place of $f'(x^k)$. Also, earlier related work along different lines concerning uniformly monotone gradient maps $f: R^n \to R^n$ was done by Bank and Rose.[5]

Theorem 2: Let f map a real Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$, *into itself such that* $\langle f(u) - f(v), u - v \rangle \geq |u - v| \psi(|u - v|)$ *for all* $u, v \in H$. *Assume that* f' *exists and is Lipschitz on bounded subsets of H. Then f is a* C^1*-diffeomorphism of H onto H, and the conclusion of Theorem* 1 *holds.*

Using Theorem 1, a proof of Theorem 2 can be obtained by trivially modifying the proof of Theorem 4 of Ref. 1.

2.3 $B = R^n$

The following complete result is a direct corollary of Theorem 1 (see the proof of Theorem 5 of Ref 1).

Theorem 3: Let $B = R^n$, *and let* f' *be Lipschitz and continuously differentiable on bounded subsets of* R^n. *Then f is a* C^1*-diffeomorphism of* R^n *onto itself if and only if*

(*i*) *Process* N_1 *can be carried out for each a and each* x^0.

(*ii*) *For each a, the sequence produced by Process* N_1 *converges quadratically to a solution x of* $f(x) = a$, *and x does not depend on* x^0.

2.4 Comments

As in Ref 1, our primary purpose is to focus attention on general relationships between diffeomorphisms and computation. Clearly, no attempt is made to optimize the performance of all aspects of the type of algorithm described. However, there are some basically self-evident modifications that are sometimes useful. For example, the total number of iterations required in a specific case can sometimes be reduced significantly by repeatedly, or occasionally, stopping the algorithm after a number of steps and resetting the initial value of λ in Process N_1 to a smaller number. (It is not difficult to give rules of thumb concerning *when* to stop the algorithm and by *how much* to reduce λ, but we have not tried to prove theorems that bear on these matters.) Of course, bounds on the location of the solution and estimates of K and Λ, which are available in some problems, can be used in an obvious way. Similarly, if for example $B = R^n$, a globally convergent steepest-descent process (see Ref. 6)* might be used initially to obtain a better approximation to the solution before the Newton-direction algorithm is used. (In fact, a well known and often useful strategy is to combine steepest descent and *pure* Newton iterations in this way.†)

* We take this opportunity to correct a typographical error in Ref. 6. On page 1004, left column, line 2, [2] should be replaced with [21].

† This paragraph was motivated by a helpful observation by D. J. Rose to the effect that, as the algorithm stands, there are cases in which many iterations are required.

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