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Characterization for Series-Parallel Channel Graphs

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Channel graphs are often used to describe the connecting paths between a pair of terminals in a link system. Series-parallel (SP) channel graphs and Takagi graphs are special classes of channel graphs heavily studied in the literature. In this paper, we give a noninductive definition of SP channel graphs which we prove to be equivalent to the usual inductive definition. We then use this new definition to characterize SP Takagi graphs and regular SP channel graphs.

I. INTRODUCTION

A graph G is called an *s-stage graph* if the set of vertices can be partitioned into s disjoint subsets $V_1, \cdots, V_s$, and the set of edges partitioned into $s-1$ disjoint subsets $E_1, \cdots, E_{s-1}$, such that edges in E_i connect vertices in V_i to vertices in V_{i+1}. G is called an *s-stage channel graph* if each of V_1 and V_s contain a single vertex, called the *source* and the *sink*, respectively, and if each vertex in V_i, $1 < i < s$, is incident to at least one edge each in E_{i-1} and E_i. For convenience, we assume that vertices in the same subset form a column and the columns are ordered from left to right in the natural way.

A channel graph is *series-parallel* (SP) if it can be obtained either by a series combination or a parallel combination of two smaller SP channel graphs, with the smallest such graph being an edge. A *series combination* of an s-stage channel graph G and a t-stage channel graph H is a horizontal union of G and H, with the last stage of G being identified with the first stage of H (thus, the new channel graph has $s + t - 1$ stages). A *parallel combination* of two s-stage channel graphs G and H is a vertical union of G and H, with the first stage of G being identified with the first stage of H, and the last stage of G being identified with the last stage of H (the new channel graph remains s-stage).

A channel graph is *regular* if for every fixed i, $i = 1, \cdots, s$, every vertex in V_i has x_i edges in E_{i-1} and y_i edges in E_i, where x_i and y_i depend only on i.

As first illustrated by Lee,[1] the blocking probability of an SP channel graph can be computed inductively. On the other hand, it is well known that the blocking probability of a nonSP channel graph can be hard to compute in general.[2] This difference in computability explains the special importance of SP channel graphs in switching network studies.

Historically, SP channel graphs have always been defined inductively. In this paper, we give a simple noninductive definition of SP channel graphs and show that it is mathematically equivalent to the inductive definition. The noninductive definition sometimes facilitates the recognition of an SP channel graph and other times it facilitates the use of its properties. Then, we apply this new definition to obtain characterizations of SP Takagi graphs and regular SP channel graphs, which are critically used in Ref. 3 for proving a result on regular SP channel graphs.

II. A NONINDUCTIVE DEFINITION OF SP CHANNEL GRAPHS

Material in this section has close analogy with the study of "series-parallel connection" and "bridge connection" in the theory of circuit design.[4,5]

Define a *bridge* (see Fig. 1) as a channel graph which consists of two paths p and q from source A to sink B that do not intersect anywhere else and a third path r from a point D on p to a point C on q.

Theorem 1: A channel graph G is SP if and only if no bridge is a subgraph of G.

Proof: We first prove that if G is SP, then G cannot contain a bridge. Our proof is by induction on the number of vertices. An edge, the smallest SP channel graph with two vertices, clearly cannot contain a bridge. Suppose G has n vertices. Since G is SP, there exist two SP channel graphs, G_1 and G_2, such that G is obtained by a series or a

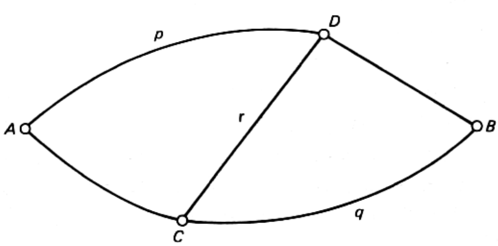

Fig. 1—A bridge channel graph.

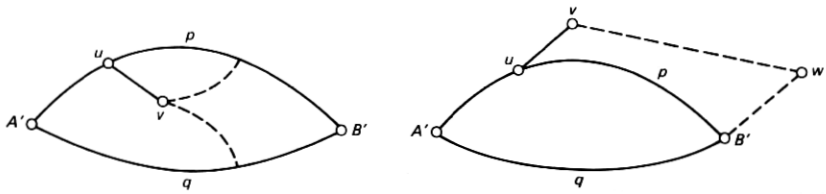

Fig. 2—Consequence of edge $[u, v]$.

parallel combination of G_1 and G_2. By the inductive assumption, neither G_1 nor G_2 contains a bridge. Furthermore, it is clear that neither a series combination nor a parallel combination of two bridgeless graphs can produce a bridge, hence, G contains no bridge.

Next we prove that if G does not contain a bridge, then G is SP. The proof is by induction on the number of edges in G. Clearly the result holds when G has one edge. For the general case, define a *circuit channel graph* as a channel graph conisting of two nonintersecting paths. Let G' be the *smallest circuit channel subgraph* of G, i.e., one containing the least number of vertices, and hence, the least number of stages and edges. Let $G \backslash G'$ denote the complementary graph of G' in G. We prove that no edge of $G \backslash G'$ includes a vertex of G', other than possibly the source or sink of G'.

Let p and q be the two paths of G' from source A' to sink B'. Suppose to the contrary that there exists an edge $[u, v]$ in $G \backslash G'$ with vertex u on p (see Fig. 2). If v is on p or q or if an extension of $[u, v]$ meets p or q, then a circuit channel graph smaller than G' exists, which is a contradiction to the definition of G'. If v is not on p or q and no extension of v intersects p or q, then since both v and B' must be connected to the sink of G, there exists a vertex w in a later stage than that of v and B' such that the two vertices v and B' are joined by paths to w, but to no such vertex in an earlier stage. However, we now see that the portion of path p from u to B' forms a bridge between two disjoint paths joining A' to w, a contradiction to the assumption made on G.

Therefore, no edge of $G \backslash G'$ includes a vertex of G' other than possibly its source or sink. Thus, the deletion of one path of G' from G certainly does not affect the SP property of G. Since G without that path contains no bridge and has fewer edges than G, the inductive assumption implies that it must be SP, hence, G must also be SP.

III. A CHARACTERIZATION OF REGULAR SP CHANNEL GRAPHS

An (i, j, r) *multiplex*, $1 \le i < j \le s$, of an s-stage channel graph G is an s-stage channel graph consisting of r copies of G where the subgraphs from stage 1 to stage i, and from stage j to stage s are

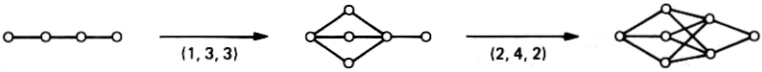

Fig. 3—Multiplexing operations.

merged into one copy (see Fig. 3). A channel graph is called a *Takagi graph* if it can be obtained by multiplexing a smaller Takagi graph where the smallest Takagi graph (with given number of stages) is a multistage line.[6–8] Corresponding to each (i, j, r) multiplex operation is a *multiplex index* $m_{ij} = r$. It has been proven (see Ref. 6) that a set of multiplex indices uniquely determines a Takagi graph, i.e., the order of the operations is not important. Let m_{ij} and m_{pq} denote two multiplex indices. Then m_{ij} is said to *cross* m_{pq} if $i < p < j < q$, and to *contain* m_{pq} if the interval $[i, j]$ contains the interval $[p, q]$. We now use Theorem 1 to give a necessary and sufficient condition that a given Takagi graph be SP.

Theorem 2: A Takagi graph is SP *if and only if it does not contain two crossing multiplex indices.*

Proof: Let G denote the given Takagi graph. Suppose G contains two crossing multiplex numbers m_{ij} and m_{pq}. Since the order of the multiplex operations is not important, we may assume the m_{ij} is the first operation and m_{pq} is the second. First, we assume $m_{ij} = m_{pq} = 2$. Then, the channel graph after each operation is as given in Fig. 4. Clearly, a bridge emerges after the second operation.

Since the graph corresponding to $m_{ij} \geq 2$ and $m_{pq} \geq 2$ contains the graph in Fig. 4b as a subgraph, our argument remains unchanged for this case. Finally, since adding other multiplex operations to the graph in Fig. 4b still results in graphs containing Fig. 4b as a subgraph, we conclude that two crossing multiplex indices always cause a bridge, hence, the graph cannot be SP.

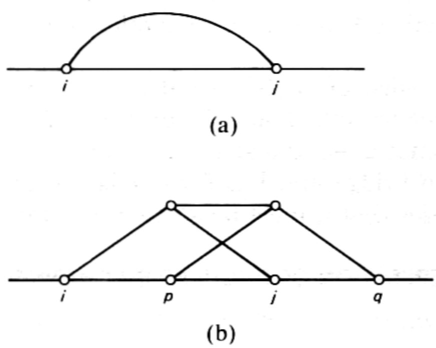

Fig. 4—The $m_{ij} = m_{pq} = 2$ Takagi graph.

Next, we prove that if G does not contain two crossing multiplex indices, then G is SP. Let m_{ij} be a multiplex index of G such that if m_{pq} is any other multiplex index of G, then either $i < p$ (hence $j \geq q$) or $i = p$ but $j > q$.

Case (*i*) $i = 1$ *and* $j = s$. Then, G is a parallel combination of m_{ij} smaller Takagi graphs. Using an induction argument, it follows that these smaller graphs are SP since they do not contain crossing multiplex indices, hence, G is SP.

Case (*ii*) $i \neq 1$. Then, G is a series combination of two smaller Takagi graphs, one from stage 1 to stage i and the other from stage i to stage s. Again by induction, these two smaller graphs are SP since they do not contain crossing multiplex indices, hence, G is SP.

Case (*iii*) $i = 1$ *but* $j < s$. Similar to Case (*ii*).

The proof is complete.

Corollary: A channel graph is regular SP if and only if it is an SP Takagi graph.

Proof: It is easily seen that every Takagi graph is regular and that every regular SP channel graph is a Takagi graph. Furthermore, a Takagi graph is a regular SP channel graph if and only if it does not contain two crossing multiplex indices. The corollary now follows immediately from Theorem 2.

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