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Digital Signal Processor:

A Tutorial Introduction to Digital Filtering

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Very-large-scale integration (VLSI) of digital electronic circuits has changed the hardware aspects of digital filters in a major way so that the use of such filters as components in commercial systems has become both economically feasible and technically desirable. Thus, large numbers of system engineers and circuit designers are now finding a need to learn about the properties of such filters, how they are used, and how they are designed. This paper is a first step toward meeting that need.

I. INTRODUCTION

The possibility of doing filtering and other signal-processing operations by numerical means instead of by traditional analog means has been known and studied for 20 years or longer. However, until recently the hardware for the physical realization of digital filters has been bulky, power-hungry, and expensive, and for this reason the digital filter has not been suitable for use as a component in commercial systems. Thus, interest in digital filters has been limited to a relatively small number of specialists doing research in this and related fields, where size, power consumption, and cost are not primary considerations.

However, VLSI has changed this condition drastically. It has reduced the size, power consumption, and cost of digital filters to the point where their use as a system component is both economically feasible and technically desirable. As a result, large numbers of system engineers and circuit designers are finding a need to learn about digital filters. Therefore, there is a place for tutorial material addressed specifically to the needs of these people.

This paper is an attempt to meet this need and addresses system engineers and circuit designers having no previous experience with digital filters or sampled-data systems. However, they are assumed to have a good understanding of the Laplace transform and its use with signals, differential equations, and electric circuits. We hope to provide a good understanding of the fundamentals of digital filtering and a strong foundation for further study of the subject. To reach these objectives most effectively, an effort is made to avoid all unnecessary abstractions. Generality is sacrificed for the sake of simplicity.

II. ELEMENTS OF DIGITAL FILTERING

This section gives an introduction to digital filtering in terms of the elementary circuit shown in Fig. 1. This simple circuit can be used to illustrate how filtering is done in the digital domain, in contrast with the more usual case of filtering in the analog domain.

The circuit is described by the following single node equation for the single unknown voltage v_2:

$$\frac{v_2 - v_1}{R} + C\frac{dv_2}{dt} = 0. \tag{1}$$

Rearranging the terms in this equation yields

$$v_2 + RC\frac{dv_2}{dt} = v_1, \tag{2}$$

which is a first-order linear differential equation in the unknown voltage v_2.

In this example, v_1 and v_2 are understood to be information bearing signals. For example, v_1 may be the output voltage of a strain gauge, the output of an accelerometer, or the output of a telephone transmitter. The information—strain, acceleration, or speech—is represented by the amplitude of v_1. The voltage v_2 represents the information after it has been processed by the RC circuit (filter). The processed form of the information, v_2, may be more valuable than the original form, v_1, because, for example, high-frequency noise has been removed from the

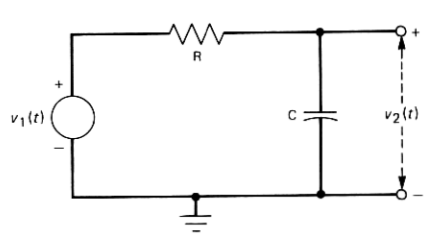


Fig. 1—An elementary filter.

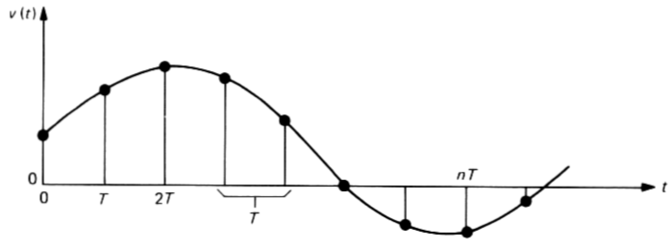


Fig. 2—Waveform of an analog signal and discrete samples of the signal.

signal. The amplitudes of the voltages v_1 and v_2 are the physical analogs of the original information—strain, acceleration, or speech—and the physical system represented by Fig. 1 is said to be an analog system.

Consider further the signal $v_2(t)$, for example. In the mathematical representation of eq. (2), voltage v_2 represents a continuum; that is, it can represent the values of any and all real numbers, and there are no real numbers that cannot be values of v_2. Thus, v_2 can change smoothly from any one value to another without any jumps or discontinuities. Similarly, time t in eq. (2) represents a continuum, and it can change smoothly from one value to another without any jumps or discontinuities. Moreover, when $v_2(t)$ represents a physical quantity, as it always does in the systems under study here, it is defined (has a numerical value) for every value of time t. These analog considerations are mentioned here because, in contrast, matters are quite different in digital filters and in digital systems.

Equation (2) can be solved easily by analytic means for the unknown voltage v_2 when the input voltage v_1 is a sinusoidal function of time, an exponential function of time, or a step function of time. It can also be solved analytically, but with more difficulty, when v_1 is a square wave and also, with still more difficulty, when v_1 is a more general periodic function of time.

When the input signal voltage in Fig. 1 is a more complicated function of time than in the few examples cited above, it is usually not practical, or even possible, to solve eq. (2) by analytic means. In such cases, however, it is possible to obtain an approximate solution by numerical methods. These numerical methods provide the basis for digital filtering. The numerical methods appropriate to this study are based on considering only discrete values of the signals, voltages v_1 and v_2 in Fig. 1, chosen at uniformly spaced instants of time. These discrete values of the signals are called samples of the signals. Figure 2 shows the waveform of a continuous analog signal voltage, and uniformly spaced samples of the signal are indicated on this diagram.

If waveforms for the analog signals v_1 and v_2 in eq. (2) exist, then sequences of samples similar to the one in Fig. 2 also exist for v_1 and v_2. Assuming the same sampling instants for the two signals, these sequences can be represented symbolically, starting at some instant designated $t = 0$, as

$$v_1(nT)\colon v_1(0),\ v_1(T),\ v_1(2T),\ \cdots,\ v_1(kT),\ \cdots,$$
$$v_2(nT)\colon v_2(0),\ v_2(T),\ v_2(2T),\ \cdots,\ v_2(kT),\ \cdots. \tag{3}$$

Now, if the time interval between samples is sufficiently small, the derivative in eq. (2) can be approximated at time $t = nT$ by

$$\frac{dv_2}{dt} \approx \frac{v_2(nT) - v_2[(n-1)T]}{T}. \tag{4}$$

As T is made smaller, the approximation becomes better. Then, if T is made sufficiently small, eq. (2) can be approximated for one instant of time, $t = nT$, by substituting eq. (4) into eq. (2) to get

$$v_2(nT) + \frac{RC}{T} v_2(nT) - \frac{RC}{T} v_2[(n-1)T] = v_1(nT). \tag{5}$$

Equation (5) is a linear difference equation that approximates the linear differential equation given by eq. (2) above for one instant of time.

The difference eq. (5) offers a possibility that is not offered by the differential eq. (2); it can be solved explicitly for the response $v_2(nT)$ to obtain

$$v_2(nT) = \frac{1}{1 + RC/T} v_1(nT) + \frac{RC/T}{1 + RC/T} v_2[(n-1)T] \tag{6}$$

$$= av_1(nT) + bv_2[(n-1)T]. \tag{7}$$

This equation gives the present sample of the response voltage $v_2(nT)$, a single sample value, in terms of the present sample of the input voltage, $v_1(nT)$, and the immediately preceding sample of the response voltage, $v_2[(n-1)T]$. If $v_1(nT)$ is a known sample of the input, and if $v_2[(n-1)T]$ is known from a previous calculation of $v_2(nT)$, then the present value of $v_2(nT)$ can be calculated by simple arithmetic from eq. (7), assuming, of course, that the coefficients a and b are known. This calculation can be repeated for successive values of $v_1(nT)$ and $v_2[(n-1)T]$ to obtain the sequence of output samples for $v_2(nT)$.

In the calculation described above, the present response sample $v_2(nT)$ depends on the present input sample $v_1(nT)$ and on the immediately preceding response sample $v_2[(n-1)T]$. Thus, eq. (7) is a kind of recursion formula in which each calculation of $v_2(nT)$ provides the value of $v_2[(n-1)T]$ for the next calculation in the sequence. The start-up behavior for this system raises an additional question, but one of no serious consequence. If the start-up instant is designated $t = 0$, which for the discrete samples corresponds to $n = 0$, then $v_1(0)$ is assumed to be known, and in addition, the value of $v_2(-T)$ is needed for the last term in eq. (7) when $n = 0$. This last requirement is similar to the requirement that the initial conditions be known when a differential equation is to be solved. If the value of $v_2(-T)$ is known from physical considerations, there is no problem. If $v_2(-T)$ is not known, a reasonable value, such as zero, can be assumed, and the sequence of calculations can begin. In this case, the first few samples of $v_2(nT)$ that are calculated will depend on the assumed initial value of $v_2[(n-1)T]$. However, the effect of the assumed initial condition decays with time, and it soon disappears.

Example 1

The simple filter of Fig. 1 is represented approximately by the difference eq. (7),

$$v_2(nT) = av_1(nT) + bv_2[(n-1)T].$$

Suppose that the input to the filter is

$$\begin{aligned} v_1(nT) &= 1, \qquad n = 0, \\ &= 0, \qquad n \neq 0, \end{aligned}$$

and suppose that

$$v_2(-T) = 0,$$

then, the response $v_2(nT)$ can be calculated as follows:

$$\begin{aligned}
v_2(0) &= a + 0 = a, && \text{for } n = 0 \\
v_2(T) &= 0 + ba = ab, && \text{for } n = 1, \\
v_2(2T) &= 0 + b(ab) = ab^2, && \text{for } n = 2, \\
v_2(3T) &= 0 + b(ab^2) = ab^3, && \text{for } n = 3, \\
&\vdots \\
v_2(nT) &= 0 + bv_2[(n-1)T] = ab^n, && \text{for } n, \\
&\vdots
\end{aligned}$$

It follows from eqs. (6) and (7) that the coefficients a and b are nonnegative and less than unity. Thus, the response $v_2(nT)$ decays with time and tends to zero as n increases without limit. This problem is treated again in Section 4.2 from a different point of view.

In the approximate representation of the circuit in Fig. 1 by the difference eq. (7), the time continuum does not exist; it has been replaced by a sequence of discrete instants of time. The sequence of samples $v_2(nT)$, for example, is a discrete-time signal, in contrast to the corresponding $v_2(t)$, which is a continuous-time signal. The discrete-time signal is defined (that is, it has a specific numerical value) only at the discrete instants of time $t = nT$, whereas the continuous-time signal is defined for every instant of time. The samples of the discrete-time signal $v_2(nT)$, for example, have finite amplitudes and zero time duration.

As shown above, the value of the present response sample $v_2(nT)$ in the circuit shown in Fig. 1 can be calculated from eq. (7) by using simple arithmetic. Only multiplication and addition (of signed numbers) are required, and it is particularly useful in this study to think of this arithmetic as being performed on a pocket-sized electronic calculator. This is true because the calculator is a digital machine that has much in common with the digital filter. All of the terms on the right-hand side of eq. (7) are entered into the calculator in digital form (decimal digits) through the keyboard, and the result of the calculation, $v_2(nT)$, is presented in digital form (decimal digits again) on the output display of the calculator.

Any given electronic calculator has a fixed number of digits in its output display, and it follows from this fact that only a finite number of discrete values can be displayed on the output. Thus, if the calculator is used to evaluate $v_2(nT)$ from eq. (7), then v_2 can no longer represent a continuum of values; it can represent only a finite number of discrete values given by the digits displayed on the output of the calculator. The information in this case is not represented by the amplitude of a physical variable, but rather, it is represented by the digits displayed by the calculator. Therefore, by definition, the information is not in analog form, and because of its form, it is called digital information. The set of digits representing the information is called a digital signal. In the case of binary systems, the information is represented by the binary-digit (bit) patterns associated with binary numbers.

Since the digital signal produced by the calculator and by the digital filter can have only a finite number of "allowed" values, the signal is quantized. One result of quantization is the introduction of random errors called quantization noise. Another result is the existence of nonlinear feedback loops in many digital filters with the likelihood of self-sustaining oscillations called limit cycles. However, these are problems associated with the design and performance of the hardware used to realize digital filters, and it is not appropriate to discuss them here. Detailed treatments of these problems are given in Refs. 1 through 3. The remainder of this paper assumes that the number of digits available for representing signals is unlimited.

In effect, sampling the analog signal shown in Fig. 2 changes the continuous-time representation of the signal to a discrete-time representation. Converting the analog-sample amplitudes in Fig. 2 to equivalent digital values changes the continuous-amplitude representation to a discrete-amplitude representation.

To summarize developments up to this point, the circuit of Fig. 1 is chosen as a simple filter to be studied for the purpose of getting an introduction to the ideas of digital filtering. The circuit is analyzed by standard techniques to obtain an analog representation in terms of the differential equation (2). Then, we imagine that the signal voltages v_1 and v_2 are sampled to obtain the difference eq. (7) as an approximation to the differential equation. Next, we envision an electronic calculator for evaluating eq. (7) by numerical methods. Data is entered into the calculator in digital form, and the result of one calculation is the value of one sample of the filter response $v_2(nT)$ in digital form. This cycle of calculation is repeated successively with successive samples of the input voltage $v_1(nT)$ to obtain successive values of the output voltage $v_2(nT)$.

Consider the case in which the input to the filter in Fig. 1 is a voice signal. To represent this signal with good accuracy by a sequence of samples, the signal must be sampled about 10,000 times per second. (The sampling is examined in detail in Section 3.2.) This fact implies that for each one-second interval of speech, about 10,000 samples of the response $v_2(nT)$ must be calculated. Although the calculation of each output sample is simple enough, calculating 10,000 of them with a manually operated calculator takes quite a while.

The stage is now set for the introduction of the digital signal processor as a means for implementing digital filters. The digital signal processor is a digital device (binary digits) that has been especially designed to perform the arithmetic required in the repetitive evaluation of the difference equation described above. It does the arithmetic automatically at very high speed under program control. When programmed to solve eq. (7) it can accept a new input signal sample in digital form, calculate the corresponding response sample, and deliver the response to the output all in less than 5 μs. Thus, the signal processor can receive a new input sample and calculate the corresponding response in a tiny fraction of the interval between successive input samples, an interval of 100 μs at 10,000 samples per second. It follows from these facts that the processor, with its blazing speed, can operate in real time, solving eq. (7) almost instantly for the response to each input sample and then waiting for the next input sample to come along. Thus, by solving eq. (7) in real time, the processor produces in sampled digital form the same response to the input signal v_1 as the filter in Fig. 1, within the accuracy permitted by the approximations involved in deriving eq. (7) and in sampling v_1.

The filter shown in Fig. 1 is an analog computer that solves the differential eq. (2) in real time. Similarly, the digital signal processor is a digital computer that solves the difference eq. (7) in real time. To the extent that eq. (7) is a good approximation to eq. (2), the processor is a good approximation to the filter in Fig. 1.

The ideas developed above make the concepts of digital filtering and the use of the digital signal processor for its realization seem to be quite simple. While the basic ideas are indeed perfectly straightforward, the implementation of high-performance filters by these means in a realistic system environment presents a challenge.

First, the question of how well difference eq. (7) approximates differential eq. (2) has been raised above. Insofar as the filtering operation is concerned, this question requires a detailed answer and a more complete mathematical formulation of the problem than we have so far given. The remainder of the paper concerns this and related problems.

Second, the simplicity of the digital filter as presented above is genuine, but in a way it is deceptive. The starting point for the presentation above is chosen to bypass all of the challenging preliminary work that is needed to put the real engineering problem into a form that can be implemented by the digital signal processor.

The linear difference equation is the central element in the concept of the digital filter. In the example represented in Fig. 1, the difference equation is obtained by approximating the differential eq. (2), which, in turn, is obtained directly from the circuit assumed in Fig. 1. However, the design of real filters rarely has this kind of starting point. For reasons which stem partly from technological heritage and partly from mathematical tractability, filter design usually starts with a specification of the frequency characteristics that are desired of the filter: Pass-band characteristics and frequencies, stop-band characteristics and frequencies, delay characteristics, and the like. Then, in order to obtain a realization in digital form of a filter having these specified characteristics, it is necessary to determine, in some way, a linear difference equation describing a filter having the specified characteristics. The digital signal processor is then programmed to evaluate this difference equation by arithmetic operations.

The most usual way of designing a digital filter is to start with classical analog-filter theory. Given a realizable set of frequency characteristics, classical theory can be used to derive the analog transfer function for an analog filter having the specified characteristics. Corresponding to this transfer function there is always a differential equation such as the one in eq. (2), for example. In principle, it would be possible to proceed as in the example in Fig. 1 and use this differential equation to derive an approximately equivalent difference equation. However, an alternative procedure proves to be more fruitful.

Every differential equation relating an output signal and its derivatives to an input signal and its derivatives gives rise, through the Laplace transform, to an analog transfer function. Similarly, as is shown in Section IV, every difference equation relating the present and past values of an output signal to the present and past values of an input signal gives rise, through the z transform, to a digital transfer function. It is shown in the remainder of this paper that the digital transfer function is related to the digital filter and its frequency characteristics in much the same way as the analog transfer function is related to the analog filter. The z transform is a special form of the Laplace transform that is developed in some detail in Section 3.1. Furthermore, we show in Section VI that an analog transfer function can be transformed into a digital transfer function in such a way that the frequency characteristics of the two functions are related in a precisely known manner. Thus, in many cases classical techniques can be used to derive a prototype analog transfer function that can be transformed into a digital transfer function having the desired frequency characteristics. The difference equation corresponding to this digital transfer function can be derived easily, as shown in Section 4.4, and it can then be implemented with hardware to obtain a physical realization of the digital filter.

The method outlined above is the most common, but not the only, method used for designing digital filters. The remainder of this paper gives the details of the method. However, the treatment is necessarily introductory, and makes no attempt to provide any expertise in the field. (See Refs. 1 through 3 for a detailed treatment of the subject.)

III. THE *z* TRANSFORM FOR SAMPLED SIGNALS

Section II introduced digital filtering in terms of a simple example. The example reveals some of the approximations involved in digital filtering, and it points out the need for a more comprehensive mathematical formulation of the problem. The z transform is the mathematical tool that is extensively used for this purpose. The objective of this section is to present the z transform and to develop its properties to the extent required by this paper.

3.1 Definition and elementary properties of the z transform

In general, digital filtering and digital signal processing concern processing signals that are characterized by a sequence of values, normally instantaneous samples of a continuous-time signal, that are uniformly spaced in time. Such signals are discussed in Section II. For the purpose of this study, we consider only signals that are zero for time t less than some instant designated $t = 0$. It is also assumed that the signal to be sampled is continuous at every sampling instant, except possibly at $t = 0$, in which case the value of the signal at $t = 0+$ is taken by convention.

Figure 2 shows a continuous-time signal and a sequence of discrete samples of that signal. The first step in the method used here for developing the z transform is to note that signal samples of zero duration cannot exist in any physical system. Any sampling operation that is implemented in hardware is necessarily associated with a holding operation that produces signal samples of nonzero duration. The most widely used sample-and-hold circuit has the form shown in Fig. 3a, and when its input voltage is the continuous-time signal voltage shown in Fig. 2, the circuit produces a stair-step output-voltage waveform shown as v_2 in Fig. 3b. It is also important to note that digital filters are often followed by a digital-to-analog converter to provide an output signal in analog form. In many cases the converter produces a stair-step output waveform like the one shown in Fig. 3b. Thus, stair-step waveforms play a central role in the analysis of digital filters and of sampled-data systems in general.

A great deal of valuable information about the sampling process and the stair-step output waveform can be obtained from the Laplace transform of the stair-step signal. This transform is

$$V_2(s) = \int_0^{\infty} v_2(t)e^{-st}dt. \tag{8}$$

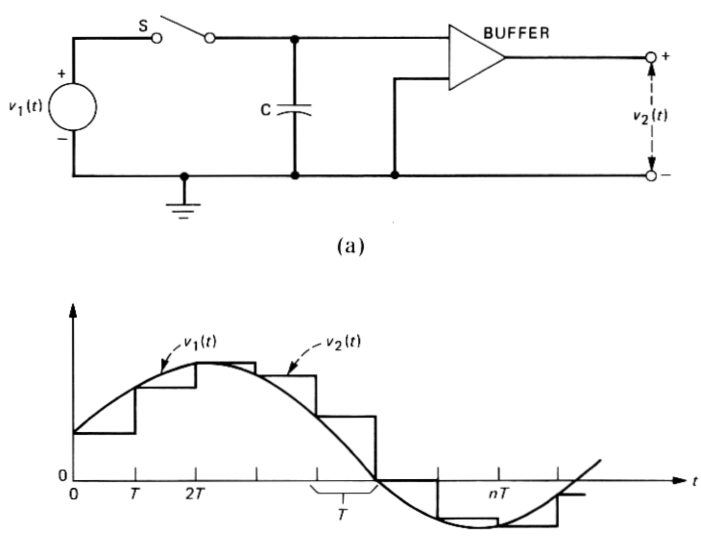


Fig. 3—Sample-and-hold circuit. (a) Circuit. (b) Waveforms.

(The convention followed in this paper is to use lower-case letters to represent instantaneous values of time varying quantities and to use capital letters for transforms and other quantities that are not functions of time.) At first glance, evaluating the integral in eq. (8) may seem to present a serious problem because of the stair-step form of $v_2(t)$. However, a little further thought shows how this problem can be reduced to the very simple problem of calculating the transform of a constant. It follows from the definition of the integral that the integral in eq. (8) can be expressed as the sum of infinitely many integrals, each spanning the time interval of one step in the waveform of $v_2(t)$. Thus, eq. (8) can be expressed as

$$V_2(s) = \int_0^T v_1(0)e^{-st}dt + \int_T^{2T} v_1(T)e^{-st}dt + \cdots + \int_{nT}^{(n+1)T} v_1(nT)e^{-st}dt + \cdots. \quad (9)$$

This equation uses the fact that during each step in the waveform, $v_2(t)$ is constant and equal to the value of $v_1(t)$ at the beginning of the step.

Equation (9) can be written more briefly as

$$V_2(s) = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} v_1(nT)e^{-st}dt. \quad (10)$$

Evaluating this integral and inserting the limits yields

$$V_2(s) = \sum_{n=0}^{\infty} v_1(nT)\left(\frac{e^{-snT} - e^{-s(n+1)T}}{s}\right). \quad (11)$$

Factoring e^{-nsT} out of the parentheses in eq. (11) produces

$$V_2(s) = \sum_{n=0}^{\infty} v_1(nT)e^{-nsT}\left(\frac{1 - e^{-sT}}{s}\right). \quad (12)$$

Now the factor in the parentheses is independent of n; hence, it can be factored out of the summation to produce

$$V_2(s) = \frac{1 - e^{-sT}}{s} \sum_{n=0}^{\infty} v_1(nT)e^{-nsT}. \quad (13)$$

The summation in eq. (13) contains the values of all the samples of the input signal $v_1(t)$, and it also contains the instant of time $t = nT$ at which each sample occurs. Thus, it contains complete information about the sequence of samples $v_1(nT)$. It is, therefore, common practice to associate the summation with the process of sampling the input signal. The factor multiplying the summation depends on the fact that the output signal $v_2(t)$ is a stair-step wave, but it is totally independent of the input signal $v_1(t)$. This factor is the same in the Laplace transform of every stair-step wave, and it is commonly associated with the hold part of the sample-and-hold operation. Since this factor is the same for every stair-step wave, it is of minor importance in the design of digital filters, although its effect must always be accounted for at some point in the design.

As a result of the relations described above, it proves to be very useful to break eq. (13) into two parts,

$$\hat{V}_1(s) = \sum_{n=0}^{\infty} v_1(nT)e^{-nsT} \tag{14}$$

and

$$H_h(s) = \frac{1 - e^{-sT}}{s}, \tag{15}$$

so that (13) can be written as

$$V_2(s) = H_h(s)\hat{V}_1(s). \tag{16}$$

The circumflex (ˆ) is used to distinguish the function in eq. (14) pertaining to the sample values of $v_1(t)$ from the Laplace transform $V_1(s)$ of the unsampled signal $v_1(t)$. The subscript h on the left side of eq. (15) signifies that H_h is associated with the hold part of the sample-and-hold operation.

The separation of eq. (13) into two parts given by eqs. (14) and (15) is a purely algebraic operation; it is not based on any consideration of any physical system. Therefore, readers should not feel frustrated if their attempts to ascribe a physical significance to these separate relations produce results that are less than completely satisfactory. However, when eqs. (14) and (15) are multiplied together, as in eq. (16), the result is always the Laplace transform of a stair-step wave in which the heights of the successive steps are equal to the values of the successive samples in eq. (14).

In eq. (14) the complex-frequency variable s appears only in the combination e^{-nsT}. Therefore, it proves to be quite convenient to define a new symbol,

$$z = e^{sT}, \tag{17}$$

so that eq. (14) can be written more simply as

$$V_1^*(z) = \sum_{n=0}^{\infty} v_1(nT)z^{-n} \tag{18}$$

$$= v_1(0) + v_1(T)z^{-1} + v_1(2T)z^{-2} + \cdots + v_1(nT)z^{-n} + \cdots. \tag{19}$$

This is the z transform of the sequence of sample values $v_1(nT)$. The transform, as given by eq. (18), is an infinite series in the variable z. Since eq. (18) is a power series, it can be shown by conventional methods that if the sequence $v_1(nT)$ is bounded, then the series converges for all values of z such that $|z| > 1$.

The relations developed above can be illustrated with the aid of Fig. 4. Figure 4a shows the beginning of a sequence of samples, $f(nT)$. The z transform of this sequence can be written by inspection with the aid of Fig. 4a; it is

$$F^*(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}. \tag{20}$$

Furthermore, a stair-step wave, $f_a(t)$, can be constructed on the samples of Fig. 4a as shown in Fig. 4b. The height of each successive step is equal to the value of each successive sample in Fig. 4a. Now the Laplace transform of the stair-step wave can be written. Using eq. (17) to replace z in eq. (20) yields

$$\hat{F}(s) = \sum_{n=0}^{\infty} f(nT)e^{-nsT}. \tag{21}$$

Then, eqs. (15) and (16) lead to

$$\begin{aligned} F_a(s) &= H_h(s)\hat{F}(s) \\ &= \frac{1 - e^{-sT}}{s} \sum_{n=0}^{\infty} f(nT)e^{-nsT}. \end{aligned} \tag{22}$$

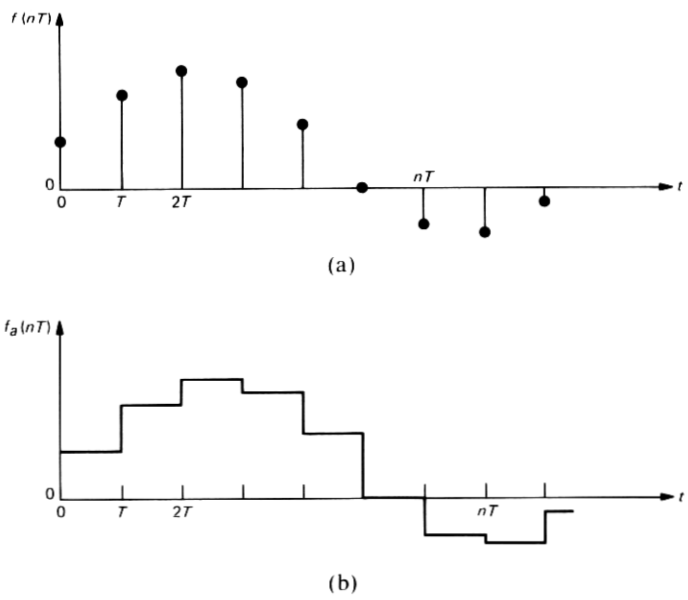


Fig. 4—Sampled data. (a) Sequence of samples. (b) Stair-step waveform.

This is the Laplace transform of the stair-step wave.

Example 2

Consider the short sequence of samples

$$f(nT)\text{: } 2, 3, 1, 0, 0, 0, \cdots.$$

The z transform of the sequence is, by inspection

$$F^*(z) = \sum_{n=0}^{2} f(nT)z^{-n} = 2 + 3z^{-1} + z^{-2}. \tag{23}$$

Using eq. (17) to replace z in eq. (23) yields

$$\hat{F}(s) = 2 + 3e^{-sT} + e^{-2sT}. \tag{24}$$

The Laplace transform of the corresponding stair-step function is given by eqs. (15) and (16) as

$$\begin{aligned} F_a(s) = H_h(s)\hat{F}(s) &= \frac{1 - e^{-sT}}{s}(2 + 3e^{-sT} + e^{-2sT}) \\ &= \frac{1}{s}(2 + 3e^{-sT} + e^{-2sT} - 2e^{-sT} - 3e^{-2sT} - e^{-3sT}) \\ &= \frac{2}{s} + \frac{1}{s}e^{-sT} - \frac{2}{s}e^{-2sT} - \frac{1}{s}e^{-3sT}. \end{aligned} \tag{25}$$

If $u(t - nT)$ represents the unit step function delayed by nT units of time, then the inverse Laplace transform of eq. (25) can be written term-by-term as

$$f_a(t) = 2u(t) + u(t - T) - 2u(t - 2T) - u(t - 3T). \tag{26}$$

As a quick sketch of eq. (26) shows, it is the stair-step wave based on the given arbitrary sequence of samples.

Example 3

Suppose that the input voltage to the sample-and-hold circuit in Fig. 3a is a unit-step function so that samples $v_1(nT)$ are unity for all nonnegative n including $n = 0$. The z transform for this sequence is, from eq. (18):

$$V_1^*(z) = \sum_{n=0}^{\infty} z^{-n}. \tag{27}$$

This series can be summed to obtain

$$V_1^*(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1. \tag{28}$$

The series in eq. (27) converges to this value for all values of z such that $|z| > 1$.

The important point in the above examples is that given an arbitrary sequence of uniformly spaced sample values, the z transform of the sequence can be written by inspection. Then, with no effort at all, the Laplace transform of the associated stair-step wave can be written, and from the Laplace transform the frequency spectrum of the wave can be calculated. The frequency spectrum is, of course, a central idea in the classical methods for the analysis and design of filters.

Similarly, given the z transform expressed as a power series in z^{-1}, the corresponding sequence of sample values can be written directly by inspection of the coefficients in the power series. In this way, the inverse of the z transform can be calculated. In the study that follows, there will be numerous occasions to write z transforms and their inverses by these simple procedures.

The z transform defined by eq. (18),

$$V_1^*(z) = \sum_{n=0}^{\infty} v_1(nT) z^{-n}, \tag{29}$$

is obtained from the Laplace transform in eq. (14),

$$\hat{V}_1(s) = \sum_{n=0}^{\infty} v_1(nT) e^{-nsT}, \tag{30}$$

by defining the symbol

$$z = e^{sT}. \tag{31}$$

The relations between z and s and between $V_1^*(z)$ and $\hat{V}_1(s)$ are important in the following study; therefore, we examine them here. For every point in the s plane, eq. (31) specifies just one point in the z plane. Conversely, however, every point in the z plane corresponds, through eq. (31), to infinitely many points in the s plane. This matter is examined in more detail later. The process by which a point in one plane is transferred to the other plane is called "mapping," and the law that governs the mapping process in the present case is eq. (31). At corresponding points in the two planes, eq. (31) is satisfied, and then eqs. (29) and (30) are equivalent with

$$V_1^*(z) = \hat{V}_1(s). \tag{32}$$

The important relationship between the z and s planes can be explored further by considering the special case in which the complex-frequency variable s takes on purely imaginary values, $s = j\omega$, and s is, thus, restricted to points in the s plane lying on the imaginary axis. The corresponding values of z are given by eq. (31) as

$$z = e^{sT} = e^{j\omega T}. \tag{33}$$

Under this condition,

$$|z| = |e^{j\omega T}| = 1, \tag{34}$$

and all values of z satisfying this relation correspond to points on the unit circle in the z plane, a circle centered at the origin and having unity radius. That is, every point on the $j\omega$ axis in the s plane corresponds to a point on the unit circle in the z plane, or, more simply, the $j\omega$ axis in the s plane maps onto the unit circle in the z plane.

It also follows from eq. (33) that

$$z = 1 \quad \text{when} \quad \omega = 0. \tag{35}$$

Furthermore, as ω increases in the positive direction from zero, the angle of z, which is just ωT rad, increases positively, and as the point s moves up the $j\omega$ axis in the s plane, the point z moves continuously in a counterclockwise direction around the unit circle in the z plane. These relations are illustrated in Figs. 5a and b. (The three parts of Fig. 5 represent the same z plane; three parts are used to avoid putting too much information into one diagram.)

Then, from eq. (33), as ω increases,

$$z = -1 \quad \text{when} \quad \omega T = \pi \text{ rad}. \tag{36}$$

If the sampling frequency is defined as

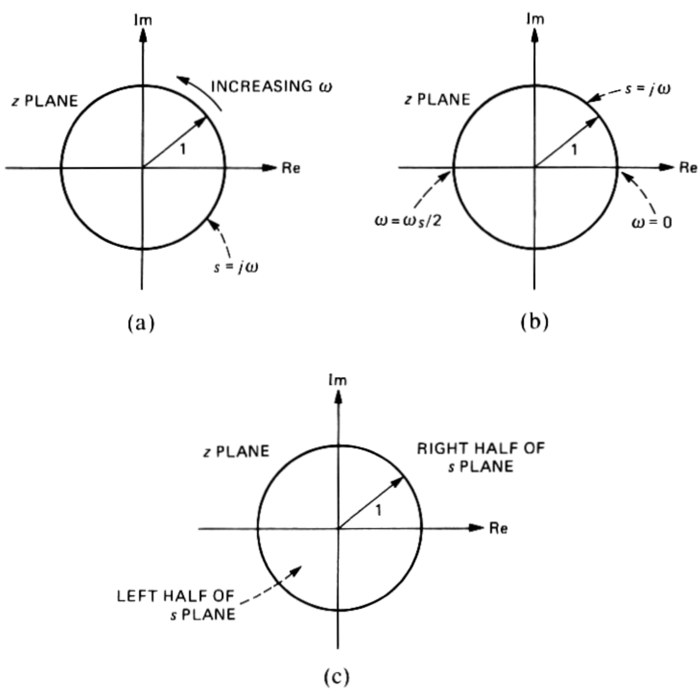


Fig. 5—Relations between the z plane and the s plane.

$$f_s = \frac{1}{T} = \frac{\omega_s}{2\pi}, \tag{37}$$

then eq. (36) can be written as

$$z = -1 \qquad \text{when} \quad \omega = \frac{\omega_s}{2} \tag{38}$$

or when

$$f = \frac{f_s}{2}. \tag{39}$$

This relation is shown in Fig. 5b. Thus, as ω increases positively from zero to $(\omega_s/2)$ rad/s, the point z moves counterclockwise around the upper semicircle in Fig. 5a.

In a similar way, as ω increases negatively from zero to $(-\omega_s/2)$ rad/s, the point z moves clockwise around the lower semicircle in Fig. 5a.

To pursue this matter further, let $s = j\omega$ move up the $j\omega$ axis without limit. It then follows from eq. (33) that the corresponding point in the z plane moves around the unit circle shown in Fig. 5 repeatedly, without limit. Thus, z passes through any given point on the unit circle an unlimited number of times as $j\omega$ increases without limit, and that single point on the unit circle in the z plane corresponds to infinitely many points uniformly spaced on the $j\omega$ axis in the s plane.

To explore this matter further, note that

$$\exp(j\omega T) = \exp[j(\omega T + 2\pi k)], \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{40}$$

and using eq. (37),

$$\exp(j\omega T) = \exp[j(\omega + k\omega_s)T], \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{41}$$

But this is the definition of a periodic function of ω with, in this case, a period equal to ω_s, the sampling frequency in radians per second. Moreover, with $s = j\omega$, eq. (30) becomes

$$\hat{V}_1(j\omega) = \sum_{n=0}^{\infty} v_1(nT)e^{-jn\omega T}, \tag{42}$$

and it follows from eq. (41) that $\hat{V}_1(j\omega)$ is also periodic with a period equal to ω_s. Thus, the frequency spectrum of the sampled wave is a periodic function of ω, a consequence of the sampling operation. We examine this further in the next section.

Returning to eq. (31) and substituting $s = \sigma + j\omega$ yields

$$z = e^{sT} = e^{\sigma T} e^{j\omega T}, \tag{43}$$

and

$$|z| = e^{\sigma T}. \tag{44}$$

Since T is always positive, it follows that

$$|z| < 1 \quad \text{when} \quad \sigma < 0. \tag{45}$$

Thus, the entire left half of the s plane is mapped by eq. (43) into the interior of the unit circle in the z plane as indicated in Fig. 5c. It is easy to show in a similar manner that the right half of the s plane is mapped into the exterior of the unit circle in the z plane. These relations are of basic importance when considering the stability (freedom from growing transients) of digital filters.

3.2 Frequency spectra of sampled signals

In Section 3.1 a continuous-time signal voltage $v_1(t)$, having a Laplace transform $V_1(s)$, is applied to the input of the sample-and-hold circuit shown in Fig. 3a. The output of this circuit is the stair-step signal voltage $v_2(t)$ shown in Fig. 3b. The Laplace transform of $v_2(t)$ is found in Section 3.1 to be

$$V_2(s) = H_h(s)\hat{V}_1(s), \tag{46}$$

where

$$H_h(s) = \frac{1 - e^{-sT}}{s}, \tag{47}$$

and

$$\hat{V}_1(s) = \sum_{n=0}^{\infty} v_1(nT)e^{-nsT}. \tag{48}$$

In Section 3.1, we show that with $s = j\omega$, $\hat{V}_1(j\omega)$ is a periodic function of ω with a period equal to ω_s, the sampling frequency in radians per second. However, the analysis given there provides no information about $\hat{V}_1(j\omega)$ beyond the fact that it is periodic. The objective here is to extend that analysis.

Since $\hat{V}_1(s)$ in eq. (48) depends on the sample values of $v_1(t)$, it seems reasonable that $\hat{V}_1(s)$ should be related in some way to $V_1(s)$, the Laplace transform of $V_1(t)$. This, in fact, is the case. It can be shown that

$$\hat{V}_1(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} V_1(s - jn\omega_s), \tag{49}$$

where, again, ω_s is the sampling frequency. The case of greatest interest is that in which $s = j\omega$, and

$$\hat{V}_1(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} V_1[j(\omega - n\omega_s)]. \tag{50}$$

This result shows, again, that $\hat{V}_1(j\omega)$ is a periodic function of ω, and it shows further that the function consists of $V_1(j\omega)$, the transform of $v_1(t)$, repeated periodically along the $j\omega$ axis with a spacing equal to ω_s.

Equation (50) is a very important relation that has been proved in the literature many times by various methods. (See Refs. 1 through 4.) All of the methods have one thing in common—they are not simple. Furthermore, the proof does not contribute any useful engineering insights; all of the results of interest to system and circuit designers are contained in the end result, eq. (50), and hence, the proof is not included here.

Equation (48) corresponds to a z transform expressed in terms of the sample values of $v_1(t)$. This expression is in a form especially suitable for time-domain studies, such as those involving difference equations and their numerical solution by digital filters. Equation (50) corresponds to the same z transform, but it is expressed in terms of $V_1(j\omega)$, the transform of $v_1(t)$. This expression is in a form especially suitable for frequency-domain studies, such as those involving the frequency characteristics of sampled signals and digital filters.

With $s = j\omega$, eqs. (46), (47), and (50) can be used to express the transform of the stair-step output voltage delivered by the sample-and-hold circuit of Fig. 3a as

$$V_2(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega T} \sum_{n=-\infty}^{\infty} V_1[j(\omega - n\omega_s)]. \tag{51}$$

This equation gives the transform of the output voltage V_2 in terms of the transform of the input voltage V_1 and a weighting factor associated with the hold part of the sample-and-hold operation. Consider the summation on the right-hand side of eq. (51). Figure 6 illustrates the important relations expressed by this summation. Figure 6a shows a possible frequency spectrum $|V_1(j\omega)|$ for the signal at the input of the circuit in Fig. 3a. Figure 6b shows a possible spectrum of the summation in eq. (51), generated by sampling the input signal. As specified by eq. (51), this spectrum is exactly the spectrum of Fig. 6a repeated periodically on the ω axis with the period ω_s.

The first thing to be noticed about the spectrum shown in Fig. 6b is that, under the conditions pictured, this spectrum contains the undistorted spectrum of Fig. 6a, the spectrum of the original signal before sampling. Thus, it contains in undistorted form all of the information in the original signal. Clearly, this is true only if the maximum frequency ω_m in the original signal is less than $\omega_s/2$, half the sampling frequency. If ω_m exceeds this limit, the periodically repeating spectra will overlap, and the information contained in the original spectrum shown in Fig. 6a will be irreversibly distorted. Distortion arising from this source is called fold-over error or, more commonly, aliasing.

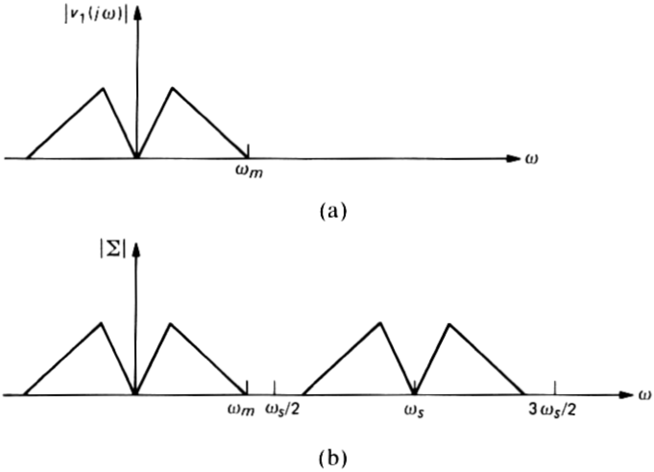


Fig. 6—Frequency spectra. (a) Spectrum of the signal before sampling. (b) Spectrum of the sum in eq. (51).

Armed with these results, it is appropriate to return now to eq. (51), the complete transform of the stair-step output voltage of the sample-and-hold circuit in Fig. 3. This equation is

$$V_2(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega T} \sum_{n=-\infty}^{\infty} V_1[j(\omega - n\omega_s)] \tag{52}$$

$$= \frac{1 - e^{-j\omega T}}{j\omega T} \{V_1(j\omega) + V_1[j(\omega \pm \omega_s)] + \cdots \}. \tag{53}$$

If the sample-and-hold circuit is designed so that there is no fold-over error, or aliasing, then it follows from Fig. 6 and eq. (53) that an ideal low-pass filter can be used to recover the spectrum of the input signal $V_1(j\omega)$ from the spectrum of the stair-step output signal $V_2(j\omega)$. The output from such a low-pass filter is, from eq. (53),

$$V_3(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega T} V_1(j\omega). \tag{54}$$

Thus, $V_1(j\omega)$ is recovered from the output of the sample-and-hold circuit, but it is weighted by another function of ω representing the filtering action of the sample-and-hold circuit. The low-pass filter used to recover $V_1(j\omega)$ is called a reconstruction filter. It is understood that $V_1(j\omega)$ is zero at all frequencies outside the passband of the reconstruction filter.

Equation (54) can be rearranged to give further insight in the following manner. Factoring $e^{-j\omega T/2}$ out of the numerator of the fraction yields

$$V_3(j\omega) = \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{j2(\omega T/2)} e^{-j\omega T/2} V_1(j\omega)$$

$$= \frac{\sin(\omega T/2)}{\omega T/2} e^{-j\omega T/2} V_1(j\omega). \tag{55}$$

Then the magnitude of $V_3(j\omega)$ is

$$|V_3(j\omega)| = \left|\frac{\sin(\omega T/2)}{\omega T/2}\right| |V_1(j\omega)|. \tag{56}$$

With the sampling frequency $f_s = 1/T$ as before,

$$\frac{\omega T}{2} = \frac{\omega}{2f_s} = \frac{\pi\omega}{\omega_s}, \tag{57}$$

and

$$|V_3(j\omega)| = \left|\frac{\sin(\pi\omega/\omega_s)}{\pi\omega/\omega_s}\right| |V_1(j\omega)|. \tag{58}$$

The first factor on the right in eq. (58) is the magnitude of the filter function associated with the sample-and-hold circuit. It is the $(\sin x)/x$ function that occurs frequently in the study of signals and the response of linear systems to signals. A plot of this function is shown in Fig. 7. As eq. (58) shows, the spectrum of the signal at the input to the sample-and-hold circuit is multiplied by this weighting curve. Figures 6 and 7 should be compared in the light of this fact. Note that the $(\sin x)/x$ weighting function in this case has zeros at ω equal to $\pm\omega_s, \pm 2\omega_s, \cdots, \pm k\omega_s, \cdots$.

In the design of precision digital filters the effects of the $(\sin x)/x$ function in Fig. 7 must be accounted for at some point in the design. In some cases, the reconstruction filter, discussed in connection with

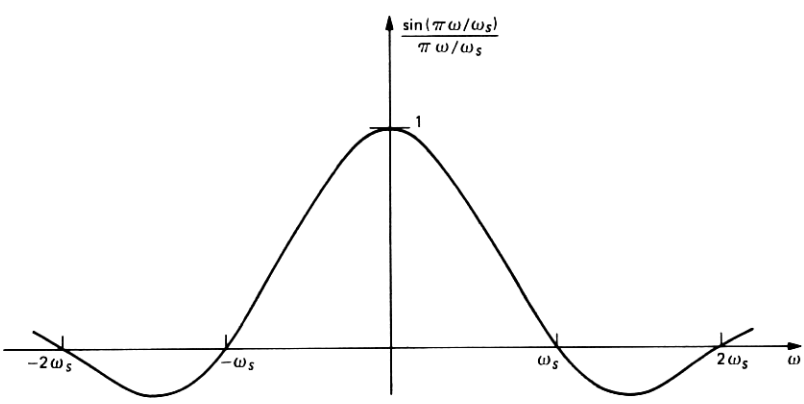


Fig. 7—Plot of the factor in eq. (58).

eq. (54), is designed to compensate for the $(\sin x)/x$ function over the band of frequencies occupied by the signal.

The developments above provide a quantitative answer to the important question of how frequently a signal must be sampled to ensure that the train of samples constitutes an accurate representation of the information contained in the signal. The results presented above show that the original information can be recovered with good accuracy if there is no fold-over error, and they also show that irremovable distortion is introduced when fold over occurs. Fold over and distortion are avoided if the maximum frequency f_m contained in the original signal is less than half the sampling frequency. Thus, for no distortion, it is required that

$$f_m < \frac{f_s}{2} = \frac{1}{2T}. \tag{59}$$

This result was first published by H. Nyquist, and the particular sampling frequency $f_s = 1/T = 2f_m$ is called the Nyquist frequency or the Nyquist rate.

From these results, it follows that the original signal must be band-limited so that a sampling frequency greater than $2f_m$ can be chosen. Furthermore, after a sampling frequency f_s has been chosen, it is still usually necessary to pass the input signal through a band-limiting filter to prevent high-frequency noise and spurious signals from being folded back on top of the baseband signal. Such band-limiting filters are usually called antialiasing filters.

It is shown above that when a signal is sampled, the information in the original signal can be recovered from the sampled signal with the aid of an ideal low-pass reconstruction filter, provided that the signal is sampled at a frequency at least twice the highest frequency contained in the original signal. In practice, however, ideal filters are not available, and practical filters require a band of frequencies in which to make the transition from the passband to the stopband. Thus, in order to avoid distortion of the information in practice, the signal must be sampled at a frequency somewhat greater than twice the maximum frequency in the original signal. Under this condition, the lobes of the periodic frequency spectrum shown in Fig. 6b are separated by an interval on the frequency axis. This interval is called a guard band, and it provides the band needed by the low-pass reconstruction filter to make the transition from the passband to the stopband.

Voice signals in telephony are usually band-limited so that the maximum frequency is in the range between 3 and 3.5 kHz, depending on the type of service involved. The sampling frequency used with such signals is usually 8 kHz, providing a guard band of from 1 to 2 kHz, depending on the maximum frequency in the signal.

IV. TRANSFER FUNCTIONS IN THE z DOMAIN

The z transform is developed in Section 3.1 as a special form of the Laplace transform with a new variable $z = e^{sT}$. The Laplace transform leads to the highly useful concept of the transfer function in the frequency domain for systems processing continuous-time signals. These transfer functions often take the desirable form of rational functions of the complex frequency s. In a similar way, the z transform leads to an equally useful transfer function in the z domain for systems processing discrete-time signals. These transfer functions often take the desirable form of rational functions of the variable $z = e^{sT}$. The objective of this section is to develop the z-domain transfer function and to examine its use in the design of digital filters.

4.1 The time-shift theorem

In the case of the Laplace transform, there are many theorems stating the mathematical properties of the transform, and similarly, there are many theorems stating basic properties of the z transform. However, among all of these z-transform theorems, there is only one that is important in this study of digital filters. This is the time-shift theorem, and it is important in deriving the z-domain transfer function from the linear difference equation (see Section II).

To develop this theorem, consider the function $f(t)$ such that

$$f(t) = 0 \qquad \text{for} \quad t < 0. \tag{60}$$

Consider also the same function delayed in time, $f(t - kT)$, where k is a positive integer and

$$f(t - kT) = 0 \qquad \text{for} \quad t - kT < 0. \tag{61}$$

The z transform of the sequence of samples representing $f(t)$ is obtained, using eq. (18), by changing t to nT and writing

$$F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n} \tag{62}$$

$$= f(0) + f(T)z^{-1} + \cdots + f(nT)z^{-n} + \cdots . \tag{63}$$

Similarly, if the delayed signal is sampled at the same instants, the transform of the delayed signal is obtained by replacing t with nT and writing

$$F_d(z) = \sum_{n=0}^{\infty} f[(n-k)T]z^{-n} \tag{64}$$

$$= f(-kT) + f[(1-k)T]z^{-1} + \cdots + f(-T)z^{-(k-1)}$$
$$+ f(0)z^{-k} + f(T)z^{-(k+1)} + f(2T)z^{-(k+2)} + \cdots . \tag{65}$$

But according to eq. (61), all terms in eq. (65) for which $n - k$ is less than zero are themselves zero, and, thus, eq. (65) reduces to

$$F_d(z) = f(0)z^{-k} + f(T)z^{-(k+1)} + f(2T)z^{-(k+2)} + \cdots$$
$$= z^{-k}[f(0) + f(T)z^{-1} + f(2T)z^{-2} + \cdots]. \quad (66)$$

Comparing eqs. (66) and (63) yields

$$F_d(z) = z^{-k}F(z). \quad (67)$$

Thus, if the z transform of the sequence $f(nT)$ is $F(z)$, then for any positive integer k, the z transform of the sequence, $f[(n - k)T]$, is $z^{-k}F(z)$.

4.2 The transfer function

The s-domain transfer function evolves from the differential equation and the Laplace transform. In a similar manner, the z-domain transfer function evolves from the difference equation and the z transform. A simple example serves as a satisfactory introduction to this topic, and such an example is provided by the circuit of Fig. 1, the corresponding difference eq. (7), and the related discussion in Section II.

The difference eq. (7) is

$$v_2(nT) = av_1(nT) + bv_2[(n - 1)T]. \quad (68)$$

This expression represents one sample value of the response $v_2(nT)$. This present value of v_2 depends on the present value of the input signal $v_1(nT)$ and also on the previous sample value of v_2. The totality of sample values of the input v_1 and the response v_2 for nonnegative values of n is given in Section II as

$$v_1(nT)\colon v_1(0), v_1(T), v_1(2T), \cdots, v_1(kT), \cdots,$$
$$v_2(nT)\colon v_2(0), v_2(T), v_2(2T), \cdots, v_2(kT), \cdots. \quad (69)$$

The z transform of $v_1(nT)$ can be written, from eq. (18), as

$$V_1(z) = \sum_{n=0}^{\infty} v_1(nT)z^{-n}, \quad (70)$$

and the z transform of $v_2(nT)$ is

$$V_2(z) = \sum_{n=0}^{\infty} v_2(nT)z^{-n}. \quad (71)$$

Similarly, eq. (68) can be transformed by multiplying by z^{-n} and summing over all nonnegative values of n; the result is

$$\sum_{n=0}^{\infty} v_2(nT)z^{-n} = a \sum_{n=0}^{\infty} v_1(nT)z^{-n} + b \sum_{n=0}^{\infty} v_2[(n - 1)T]z^{-n}. \quad (72)$$

Now, using eqs. (70) and (71) together with the time-shift theorem (eqs. (64) and (67)), eq. (72) can be written simply as

$$V_2(z) = aV_1(z) + bz^{-1}V_2(z). \tag{73}$$

Equation (73) can now be solved to obtain the response $V_2(z)$ as an explicit function of the input $V_1(z)$, and the result is

$$V_2(z) = \frac{a}{1 - bz^{-1}} V_1(z). \tag{74}$$

The z-domain transfer function is then defined as

$$H(z) = \frac{V_2(z)}{V_1(z)} = \frac{a}{1 - bz^{-1}}. \tag{75}$$

In this way, given a linear difference equation, the corresponding z-domain transfer function can be determined. The result is a rational function of the variable z, and the coefficients in the transfer function are the coefficients in the difference equation. For digital filters the coefficients are normally real numbers. Of much greater importance to the design of digital filters, however, is the fact that this process works equally well in the opposite direction. Given a z-domain transfer function that is a rational function of z with real coefficients, a corresponding linear difference equation can be determined. This reverse operation is of basic importance because the digital filter performs the filtering function by the numerical evaluation of the difference equation, as discussed in some detail in Section II. The required difference equation is almost always obtained from a z-domain transfer function.

Of further importance in this connection is the fact that there are many ways in which s-domain transfer functions for continuous-time signals can be transformed into approximately equivalent z-domain transfer functions for discrete-time signals. Thus, frequency-domain specifications for a filter can be used with classical procedures to obtain an s-domain transfer function, the s-domain transfer function can then be transformed into a z-domain transfer function, and finally, the difference equation required by the digital filter can be determined from the z-domain transfer function. These matters are discussed in more detail in Section VI.

The term *z-domain transfer function* is often replaced by the term *digital transfer function* to emphasize the fact that $H(z)$ is associated with a digital filter. Similarly, to preserve the distinctions, the s-domain transfer function is often called an *analog transfer function.* These terms are used in the remainder of this paper.

Returning to eq. (74) and using the definition in eq. (75), the transform of the response can be written as

$$V_2(z) = \frac{a}{1 - bz^{-1}} V_1(z) = H(z)V_1(z). \tag{76}$$

Some further insights can be gained by considering the special case in which the input signal has a z transform $V_1(z) = 1$. This transform corresponds to a sample sequence $v_1(nT)$ that has a value of unity for $n = 0$ and a value of zero for all other values of n. Such a signal is sometimes called a unit-sample signal. With this input, eq. (76) becomes

$$V_2(z) = H(z) = \frac{a}{1 - bz^{-1}}, \tag{77}$$

and it represents the unit-sample response of the difference equation (68). (For future use, note the similarity between the unit-sample response in this case and the unit-impulse response in the case of the analog transfer function.) The inverse transform of eq. (77) gives the time-domain response $v_2(nT)$ to the unit-sample input. The inversion can be performed by expanding the right-hand side of eq. (77) in a power series in z^{-1} and then reading off the coefficients of successive powers of z^{-1}. One way to get such a power series in this case is to perform algebraically the division indicated by eq. (77); when this is done, the result is

$$V_2(z) = H(z) = a + abz^{-1} + ab^2z^{-2} + \cdots. \tag{78}$$

The coefficients in the successive terms of this series are the uniformly spaced samples of the unit-sample response $v_2(nT)$; they are

$$v_2(nT)\colon a, ab, ab^2, \cdots, ab^n, \cdots. \tag{79}$$

Note that exactly this result is obtained in Example 1 of Section II by direct evaluation of difference eq. (7).

The sample values given in eq. (79) lie on a decaying exponential curve having the equation

$$v_2(t) = ab^{t/T}. \tag{80}$$

This assertion can be verified by substituting $t = nT$ and comparing the result with the general term in the sequence given by expression (79). It follows from eqs. (6) and (7) that b is nonnegative and less than unity. Thus, $v_2(t)$ in eq. (80) decreases by the factor b (less than unity) in every interval of duration T, which is in agreement with eq. (79).

The sequence of sample values given in expression (79) constitutes the inverse transform of the digital transfer function for the filter in Fig. 1, and it is also the response of the filter to a unit-sample input. In the study of the Laplace transform, the inverse transform of the analog transfer function is identified with the response of the filter to a unit-impulse input. The terminology developed for the analog transfer function is carried over to the digital transfer function, and the inverse of the digital transfer function is also called the impulse response of the filter, although in fact it is the unit-sample response. Thus expression (79) is called the impulse response of the digital transfer function in eq. (77). Note that the inverse transform given by expression (79) consists of a sequence of infinitely many samples. Many digital filters have impulse responses of this form, and as a class they are called infinite-impulse-response (IIR) filters.

The digital transfer functions that are useful in the design of digital filters are usually rational functions of z that can be written in the form

$$H(z) = \frac{N(z)}{D(z)}, \tag{81}$$

where N and D are polynomials in z. The impulse response (unit-sample response) associated with these functions can be determined by writing N and D in ascending powers of z^{-1} and performing the indicated division, as was done to obtain eq. (78). In general, the impulse response is a sequence of infinitely many samples as in eq. (78). However, there is a subset having the form

$$H(z) = N(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}, \tag{82}$$

and members of this subset are sometimes used as the basis for digital filters. In this case, the impulse response has a finite number of terms, $a_0, a_1, \cdots a_N$, given by eq. (82), and no division is needed to determine them. Filters of this class are called finite-impulse-response (FIR) filters. These filters are the digital counterparts of the classical analog transversal filters realized with the aid of electrical delay lines. See Refs. 1 through 3 for further information on FIR filters.

4.3 Frequency characteristics of the digital transfer function

Section 4.2 shows how the digital transfer function can be developed from a linear difference equation. The main result is given by eqs. (74) and (75), and it has the form

$$V_b(z) = H(z) V_a(z), \tag{83}$$

where $H(z)$ is the digital transfer function and V_a and V_b are transforms of the input and output signals, respectively. At this point, it is possible to examine to some extent how the transfer function performs as a filter and to gain some understanding of its frequency characteristics and how they affect the transmission of signals.

This study is based on the results developed in Section III; therefore, it is helpful to revert to the symbolism used in eqs. (14) through (19). Thus, eq. (83) is rewritten as

$$V_b^*(z) = H(z) V_a^*(z). \tag{84}$$

The transform $V_b^*(z)$ represents a sequence of discrete-time sample values $v_b(nT)$. However, filter characteristics, as understood by engineers, have meaning only in terms of continuous-time analog input and output signals. Therefore, it is assumed here that the digital transfer function (filter) is followed by a digital-to-analog converter that generates a stair-step wave based on the sequence of sample values $v_b(nT)$, as illustrated in Fig. 4, to produce an output in analog form.

The Laplace transform of this stair-step wave is given by eq. (16) as

$$V_c(s) = H_h(s) \hat{V}_b(s). \tag{85}$$

The next step concerns certain definitions that are made in Section 3.1. The identities below are not functional relations—they are pure definitions. The quantity $V_b^*(z)$ in eq. (84) is defined by eqs. (14), (17), and (18) to be

$$\sum_{n=0}^{\infty} v_b(nT) e^{-nsT} \equiv \hat{V}_b(s) \equiv V_b^*(e^{sT}) \equiv V_b^*(z), \tag{86}$$

and similarly, by definition,

$$\sum_{n=0}^{\infty} v_a(nT) e^{-nsT} \equiv \hat{V}_a(s) \equiv V_a^*(e^{sT}) \equiv V_a^*(z). \tag{87}$$

Now, substituting $z = e^{sT}$ into eq. (84) and using eq. (86) leads to

$$V_b^*(e^{sT}) = H(e^{sT}) V_a^*(e^{sT}) = \hat{V}_b(s). \tag{88}$$

Rearranging this equation and using (87) produces

$$\hat{V}_b(s) = H(e^{sT}) V_a^*(e^{sT}) = H(e^{sT}) \hat{V}_a(s). \tag{89}$$

This result can now be substituted into eq. (85) to obtain

$$V_c(s) = H_h(s) H(e^{sT}) \hat{V}_a(s), \tag{90}$$

for the transform of the stair-step output.

For $s = j\omega$, eq. (90) becomes

$$V_c(j\omega) = H_h(j\omega) H(e^{j\omega T}) \hat{V}_a(j\omega), \tag{91}$$

and substituting eq. (50) for $\hat{V}_a(j\omega)$ yields

$$V_c(j\omega) = \frac{1}{T} H_h(j\omega) H(e^{j\omega T}) \sum_{n=-\infty}^{\infty} V_a[j(\omega - n\omega_s)]. \tag{92}$$

This is the transform of the stair-step output of the digital filter followed by a digital-to-analog converter.

To interpret eq. (92), it is best to start at the far right and work back to the left. The quantity $V_a(j\omega)$ is the transform of the analog signal at the input to the filter before it is sampled. The offset $n\omega_s$ and the summation represent the periodic frequency spectrum generated by sampling the input signal; an example of such a spectrum is shown in Fig. 6. The factor $H(e^{j\omega T})$ multiplying the summation in eq. (92) is the digital transfer function expressed as a function of $s = j\omega$. The magnitude and phase characteristics of this factor modify the frequency spectrum of the sampled input signal in the usual way, and by this process the filter performs its function. The design of digital filters is concerned mainly with this factor. The factor $H_h(j\omega)$ is given by eq. (15) with $s = j\omega$. This factor, together with the multiplier $1/T$, contributes the $(\sin x)/x$ weighting function shown in eq. (58) and Fig. 7.

In summary, the frequency characteristics of the digital filter followed by a digital-to-analog converter are obtained [apart from the $(\sin x)/x$ factor] from the digital transfer function $H(z)$ with $z = e^{j\omega T}$. This transfer function is, of course, closely related to the linear difference equation from which it is derived and which is implemented by the digital filter.

4.4 Difference equations from the transfer function

Section 4.2 shows how the digital transfer function can be determined from a given difference equation. This section is concerned with the inverse operation, determining the difference equation from a given digital transfer function. The importance of this operation lies in the fact that the digital filter operates, as described in Section II, by calculating values given by a difference equation. However, filter specifications are usually given in terms of frequency characteristics, and such specifications lead to transfer functions. Thus, the required difference equation is usually obtained from a digital transfer function.

The transfer functions that are appropriate for this study are rational functions of z with real coefficients, expressed as

$$H(z) = \frac{N(z)}{D(z)},$$

where N and D are polynomials in z. In the study of these functions, we note that if the polynomials N and/or D are of degree greater than two, and especially if the roots of N or D are located close together in the z plane, then practical realization of the corresponding digital filter often proves to be unsatisfactory. The performance of the filter is so sensitive to the values of the coefficients in N and D that high-quality performance cannot be obtained. Therefore, special procedures must be followed to make possible the realization of precision digital filters of high complexity. One special procedure that can be followed is to decompose the complex filter into a cascade of noninteracting biquadratic (second-order) sections, plus possibly a single first-order section. This procedure may not be the best way to solve the problem, but it is simple, it always works, and, therefore, it is widely used. (It is noted in passing that exactly the same problem arises in the design of active RC analog filters, and it is often solved in the same way.) Thus, in the remainder of this paper, the most complex transfer function to be considered, a basic building block, is the biquadratic function of the form

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}. \tag{93}$$

If the response of the sampled-signal system corresponding to eq. (93) is designated $r(nT)$ with the z transform $R(z)$, and if the stimulus (input) is designated $s(nT)$ with the transform $S(z)$, then eq. (93) can be used to write

$$R(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} S(z). \tag{94}$$

At this point, it is appropriate to note for future use that eq. (94) can be written in the alternative form

$$R(z) = a_0 \frac{1 + a_1' z^{-1} + a_2' z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} S(z). \tag{95}$$

When several such sections are connected in cascade, it may be possible to lump some or all of the scale factors a_0 into a single scale factor for the entire filter. The result is a reduction in the number of multiplications that the hardware must perform, a fact that is discussed further in Section V. However, a potential problem exists in this connection, because the signal level throughout the filter must be kept in a suitable range. If the signal level is too large, some register in the filter will overflow and cause distortion. If the signal level is too small, the signal-to-noise ratio will suffer. Thus, it may be necessary to distribute the scale factors throughout the filter.

Equation (94) can be rearranged algebraically to obtain

$$R(z) = a_0 S(z) + a_1 z^{-1} S(z) + a_2 z^{-2} S(z) - b_1 z^{-1} R(z) - b_2 z^{-2} R(z). \tag{96}$$

From the definition of the z transform in eq. (18), it follows that

$$S(z) = \sum_n s(nT) z^{-n}, \tag{97}$$

where it is understood that the summation is over all nonnegative values of n. Similarly, the time-shift theorem, eqs. (64) and (67), yields

$$z^{-1}S(z) = \sum_n s[(n-1)T]z^{-n}, \qquad (98)$$

and

$$z^{-2}S(z) = \sum_n s[(n-2)T]z^{-n}. \qquad (99)$$

Using equivalences of this kind throughout eq. (96) yields

$$\sum_n r(nT)z^{-n} = \sum_n a_0 s(nT)z^{-n} + \sum_n a_1 s[(n-1)T]z^{-n} + \sum_n a_2 s[(n-2)T]z^{-n} - \sum_n b_1 r[(n-1)T]z^{-n} - \sum_n b_2 r[(n-2)T]z^{-n}. \qquad (100)$$

The desired difference equation is the inverse of this transform equation. The inversion is performed by removing the summations and cancelling the common factor z^{-n}; the result is

$$r(nT) = a_0 s(nT) + a_1 s[(n-1)T] + a_2 s[(n-2)T] - b_1 r[(n-1)T] - b_2 r[(n-2)T]. \qquad (101)$$

The digital filter can be programmed to perform, in real time, the multiplications and additions of signal samples and coefficients required to evaluate the right-hand side of eq. (101) for each successive sample of the response $r(nT)$. A simple example of this operation is presented in Section II. According to eq. (92) and the related discussion, the frequency characteristics of the resulting digital filter are given by $H(e^{j\omega T})$, the transfer function of eq. (93) with $z = e^{j\omega T}$.

Each specific biquadratic filter section is characterized completely for the filter hardware by the coefficients a_j and b_k of the difference eq. (101). (The quantities r and s are time varying data values.) These coefficients are typically part of the program for the filter, and they are stored with the program in the read-only memory (ROM) of the hardware. It is also significant to note that the coefficients in eq. (101) are identical with the coefficients in the digital transfer function of eq. (93). Thus, when the digital transfer function for the biquadratic section has been determined, the design is complete, and the programming of the signal processor can begin.

The present value of the response $r(nT)$ in eq. (101) depends on both the present and past values of the stimulus and also on past values of the response. Or, stated another way, the response $r(nT)$ calculated at the present will be used in calculating future values of the response. Thus, eq. (101) is a kind of recursion formula for calculating successive values of the response. For that reason, filters of this class are called recursive filters. Note that the transfer function for the FIR filter, given by eq. (82), has all its b coefficients equal to zero. Thus, the response of the FIR filter does not depend on past values of the response, and hence, there is no recursion. Filters of this class are called nonrecursive filters.

V. NUMERICAL SOLUTION OF DIFFERENCE EQUATIONS

Any biquadratic digital filter is described by a linear difference equation of the form given by eq. (101). The a_j and b_k on the right-hand side of this equation are coefficients of the difference equation (filter), and they are usually stored in the ROM of the signal processor. The quantities r and s are either past data values stored in the random-access memory (RAM) of the processor or present data values available at the terminals of the processor. The filter operates by performing the multiplications and additions required to evaluate the right-hand side of eq. (101) and, thereby, determining each successive value of the response $r(nT)$. The result is sent to the output and also stored in RAM as a new past value of data. When each new value of input sample $s(nT)$ comes in, often at the 8-kHz rate discussed in Section 3.2, the processor performs the arithmetic mentioned above and produces a new value of response $r(nT)$ in a few microseconds. The processor then either waits for a new input sample, or, more commonly, it is assigned other tasks to perform until a new sample arrives.

A flow diagram for the calculations described above is shown in Fig. 8. Note that, although the symbols look like hardware blocks, this is not a hardware block diagram; it is much more closely related to the flow chart used to diagram computer programs. The nodes in this diagram labeled $s[(n-1)T]$, $r[(n-1)T]$, etc., represent RAM storage of past data values, and the blocks labeled D represent time delays of

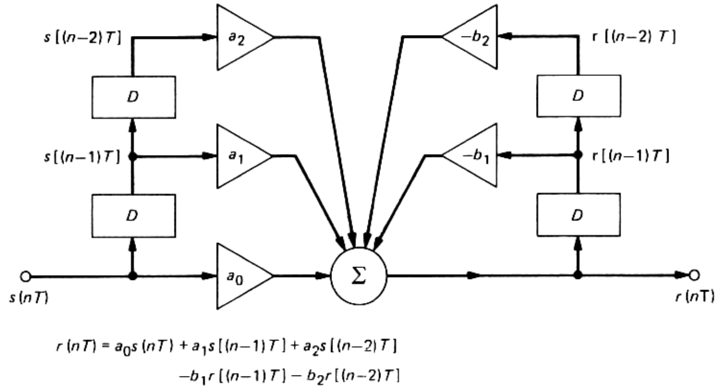


Fig. 8—Flow diagram of the calculations for a biquadratic filter. The symbol D represents a time delay of T units, one sampling interval.

T units, one sampling period. The triangles labeled a_j and b_k represent multiplications of the indicated data values by the filter coefficients.

The computational procedure pictured in Fig. 8 requires four RAM locations for the storage of past data values. This memory requirement can be cut in half by rearranging the computational procedure. One way to develop the modified procedure is to return to the transform relation in eq. (94),

$$R(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} S(z). \qquad (102)$$

This equation can be separated into two parts by defining a new, intermediate, variable given by

$$S_m(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}} S(z), \qquad (103)$$

$$= S(z) - b_1 z^{-1} S_m(z) - b_2 z^{-2} S_m(z). \qquad (104)$$

The quantity $S_m(z)$ represents a modified form of the input stimulus. Now substituting eq. (103) into eq. (102) yields

$$R(z) = (a_0 + a_1 z^{-1} + a_2 z^{-2}) S_m(z), \qquad (105)$$

$$= a_0 S_m(z) + a_1 z^{-1} S_m(z) + a_2 z^{-2} S_m(z). \qquad (106)$$

Repeating the procedure followed in eqs. (96) through (101), eqs. (104) and (106) yields the following two simultaneous difference equations for the biquadratic filter section:

$$s_m(nT) = s(nT) - b_1 s_m[(n-1)T] - b_2 s_m[(n-2)T], \qquad (107)$$

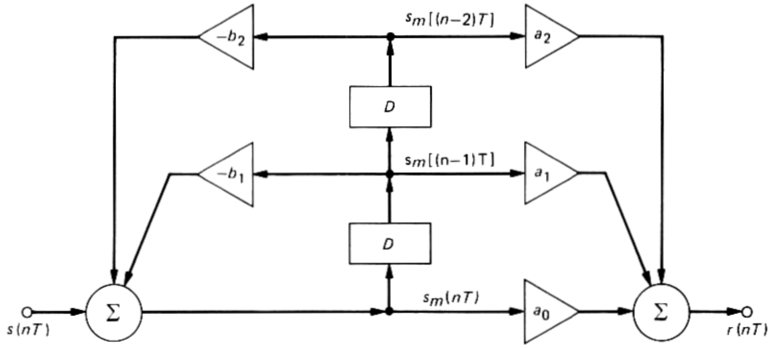
and

$$r(nT) = a_0 s_m(nT) + a_1 s_m[(n-1)T] + a_2 s_m[(n-2)T]. \qquad (108)$$

For each successive sample of input stimulus $s(nT)$, these two equations yield the corresponding sample of response $r(nT)$. Equations (107) and (108) together produce the same result as eq. (101). In this case, however, only two quantities, $s_m[(n-1)T]$ and $s_m[(n-2)T]$, need to be stored in RAM.

A flow diagram for the solution of eqs. (107) and (108) is shown in Fig. 9. As in the case of Fig. 8, this is not a hardware block diagram; instead, it is a flow diagram intended to help the reader visualize the computational steps involved in the solution of eqs. (107) and (108). The symbols used in Fig. 9 are the same as those used in Fig. 8.

The five triangular blocks in Fig. 9 represent five multiplications used to evaluate one sample of the response $r(nT)$. In this connection, it is appropriate to return briefly to eq. (95) and note that, if the scale factor is to be accounted for at a later point, the coefficient a_0 in eq.



$$s_m(nT) = s(nT) - b_1 s_m[(n-1)T] - b_2 s_m[(n-2)T]$$

$$r(nT) = a_0 s_m(nT) + a_1 s_m[(n-1)T] + a_2 s_m[(n-2)T]$$

Fig. 9—Flow diagram of the alternative calculations for a biquadratic filter. The symbol D represents a time delay of T units, one sampling interval.

(95) has been normalized to unity. But Fig. 9 shows that if a_0 is unity, then no multiplication by a_0 is needed; a direct transmission of $s_m(nT)$ to the output is sufficient. In this case, the system becomes what is sometimes called a 4-multiply biquadratic section. The 4-multiply section uses less ROM and is somewhat faster than the 5-multiply section.

When power is first applied to the digital filter, the RAM contains random values for the past data values $s_m[(n-1)T]$ and $s_m[(n-2)T]$, and the first few samples of the response $r(nT)$ that are calculated depend on these random initial conditions. However, the effects of the random initial conditions decay with time, and they soon disappear.

VI. CONSTRUCTING THE DIGITAL TRANSFER FUNCTION H(z)

In the preceding sections we showed how the digital filter operates by the numerical evaluation of the appropriate linear difference equation and how the appropriate difference equation can be obtained from the appropriate digital transfer function, a function related to the difference equation by the z transform. The problem that remains, and which the following paragraphs address, is finding the appropriate digital transfer function. No attempt is made to give a complete treatment of this challenging and multifaceted problem. Instead, attention is focused on a single technique that is possibly the most widely used solution. (See Refs. 1 through 3 for further discussions on this technique and others.)

6.1 General considerations

The design of filters, both analog and digital, usually begins with a set of specifications in the frequency domain. These specifications are usually concerned with the passband of the filter, the stopband, attenuation peaks, delay characteristics, and similar frequency-domain considerations. Given this information, there are extensive theoretical tools and computer aids that can be used to obtain a continuous-time analog filter meeting the specifications, provided of course that the specifications are physically realizable. In most cases, the design of discrete-time digital filters takes full advantage of these design aids by first producing a suitable prototype analog filter. Then the s-domain transfer function for the analog filter is transformed into a z-domain transfer function for a digital filter that provides the desired frequency characteristics.

The end result required is a linear difference equation with real coefficients. This result is to be obtained from a digital transfer function by the procedure presented in Section 4.4, eqs. (96) through (101). To obtain the desired result by this method, the transfer function must be a rational function of z, and it must have real coefficients.

The transformation of the prototype analog transfer function into a suitable digital transfer function can be accomplished with the aid of a relation, called a transformation, having the general form

$$s = F(z). \tag{109}$$

When $F(z)$ is properly chosen, substituting it for s in the prototype analog transfer function produces the desired digital transfer function. The problem now is to find a suitable transformation $F(z)$.

But first some related matters need consideration. One of these is the fact that two transformations are now involved in the design of a digital filter. The first is eq. (109), used to obtain a digital transfer function $H(z)$. Then, as shown in connection with eqs. (90) and (92), the frequency characteristics of the digital filter are determined by substituting the transformation

$$z = e^{sT} \tag{110}$$

for z in $H(z)$. Both of these transformations are satisfied independently, and both of them involve the complex-frequency variables s, although each in a different context. In order to avoid confusing these independent uses of s, it is helpful to rewrite eqs. (109) and (110) as

$$s_a = F(z) \tag{111}$$

and

$$z = \exp(s_d T), \tag{112}$$

where the subscripts a and d designate analog and digital, respectively. This notation leads to further symbolism as follows:

$H_a(s_a)$ represents the prototype analog transfer function.
$H(z)$ represents the transformed function, a digital transfer function.
$H_d(s_d)$ represents the transformed function with z replaced by $\exp(s_d T)$. This function gives the frequency characteristics of the digital filter.

An important feature of transformations, such as eqs. (111) and (112), can be illustrated in the following way. Consider first a specific value of the analog frequency

$$s_a = s_{a1}. \tag{113}$$

For this value of s_a, the analog transfer function has a specific value, a complex number,

$$H_a(s_{a1}) = X + jY. \tag{114}$$

The corresponding value of z is obtained by solving eq. (111),

$$s_{a1} = F(z_1). \tag{115}$$

The digital transfer function $H(z)$ is obtained from $H_a(s_{a1})$ by replacing s_{a1} with the equal number $F(z_1)$. Thus,

$$H(z_1) = H_a(s_{a1}) = X + jY. \tag{116}$$

Hence, values of s_a and z that satisfy eq. (111) produce values of $H_a(s_a)$ and $H(z)$ that are equal.

In exactly the same way, values of z and s_d that satisfy eq. (112) produce values of $H(z)$ and $H_d(s_d)$ that are equal, and hence,

$$H_d(s_{d1}) = H(z_1) = X + jY, \tag{117}$$

where X and Y have the same values as in eqs. (114) and (116).

In summary,

$$H_a(s_{a1}) = H(z_1) = H_d(s_{d1}) = X + jY, \tag{118}$$

provided that s_{a1}, z_1, and s_{d1} are related by the transformations of eqs. (111) and (112). Thus, if $H_a(s_a)$ is known for various values of s_a, the the values of $H(z)$ and $H_d(s_d)$ are known for the corresponding values of z and s_d. This fact is important because it relates the frequency characteristics of the digital filter, $H_d(s_d)$, to those of the prototype analog filter, $H_a(s_a)$.

These relations also hold when the transfer functions are infinite; thus, if $H_a(s_a)$ has a pole at s_{a1}, then $H(z)$ and $H_d(s_d)$ have poles at the corresponding points z_1 and s_{d1}.

6.2 The bilinear transformation

With the preliminaries taken care of, it is now appropriate to return to the main problems of identifying a suitable transformation $F(z)$ for use in eq. (111). It seems that the most efficient way of presenting this subject is simply to state at the outset a transformation that does the job well in many cases. The properties of this transformation can then be developed, and the significance of these properties can be examined to gain useful insight into the nature of the problem. In this way, a general understanding of the problem can be gained.

A transformation that does the job well, that is simple, and that is possibly the transformation most widely used in the design of digital filters is one that causes eq. (111) to take the form

$$s_a = \frac{2}{T}\frac{z-1}{z+1} = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}, \tag{119}$$

where T is the sampling interval. Since this transformation is the ratio of two linear polynomials in z, it is called a *bilinear* transformation. The multiplier $2/T$ is simply a scale factor, and it proves to be a particularly convenient one. This choice is not a requirement, however, and various writers on this subject use various scale factors. Nevertheless, the final result always comes out the same, for in every case a frequency-scale adjustment, described in Section 6.3, is required, and this adjustment compensates exactly for the various values used for the scale factor in eq. (119).

The use of this transformation always yields a digital filter with an IIR (see Section 4.2). For those cases in which a FIR filter is desired, a different procedure must be used. (See Refs. 1 through 3 for further information on FIR filter design.)

The bilinear transformation has been studied in detail by mathematicians (see Ref. 5), and it has been used to advantage in various applications in physics and engineering. For example, it is the basis for the Smith chart, a valuable tool in the study of transmission lines. Each value of z produces only one value of s_a; therefore, the transformation is single valued. The inverse of the transformation,

$$z = \frac{1 + s_a T/2}{1 - s_a T/2}, \tag{120}$$

is also bilinear, and hence, it is also single valued. This is the most general transformation that is single valued in both directions. Note that the transformation of eq. (112) is multivalued in the inverse direction, as is shown in Section 3.1.

The circle is a characteristic figure for the bilinear transformation. If straight lines are treated as circles of infinite radius, then, under this transformation, all circles in one plane transform into circles in the other plane. For example, the rectangular grid lines for the real and imaginary parts of s_a transform into sets of orthogonal circles in the z

plane with every circle passing through the point $z = -1$. As shown in Section 6.3, one of these grid lines, the $j\omega_a$ axis, transforms onto the unit circle in the z plane. This fact is of considerable importance because the other transformation used in this work, eq. (112), transforms the $j\omega_d$ axis of the s_d plane onto the unit circle in the z plane. Thus, the unit circle in the z plane serves as a bridge between the $j\omega_a$ axis and the $j\omega_d$ axis, and hence it is important in relating the frequency characteristics of the prototype analog filter to those of the digital filter.

The significant properties of the bilinear transformation, developed in Section 6.3, are listed here for immediate use. They are as follows:

(*i*) Rational functions of s_a with real coefficients are transformed into rational functions of z with real coefficients. This property ensures that the $H(z)$ obtained will produce a usable difference equation.

(*ii*) The left half of the s_a plane is mapped into the interior of the unit circle in the z plane. This property ensures that every stable prototype analog filter will lead to a stable digital filter.

(*iii*) The $j\omega_a$ axis of the s_a plane is mapped onto the unit circle in the z plane. This property ensures that the frequency characteristics of the analog filter, $H_a(j\omega_a)$, are preserved in the digital filter, $H_d(j\omega_d)$, although in general there will be some warping of the frequency scale. See eq. (118).

The two transformations involved in the design of digital filters are repeated here for convenience. They are

$$s_a = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}, \tag{121}$$

and

$$z = \exp(s_d T). \tag{122}$$

Consider the stability of the digital filter. It follows from eq. (90) and the related discussion that if the digital filter is to be stable, the poles of the digital transfer function, $H[\exp(s_d T)] = H_d(s_d)$, must lie inside the left half of the s_d plane. Furthermore, as Section 3.1 shows, the transformation given by eq. (122) maps the entire left half of the s_d plane into the interior of the unit circle in the z plane, and it maps the right half of the s_d plane into the exterior of the unit circle. Thus, for a stable digital filter the poles of the corresponding $H(z)$ must lie inside the unit circle. But, as stated in item (*ii*) above, the bilinear transformation in eq. (121) maps the entire left half of the s_a plane into the interior of the unit circle. Therefore, with this transformation every pole in the left half of the s_a plane maps through the unit circle into the left half of the s_d plane, and every stable prototype analog filter leads to a stable digital filter. This is a very desirable feature.

The bilinear transformation maps the poles and zeros anywhere in the left half of the s_a plane, suitable for stable analog filters, into the interior of the unit circle in the z plane, suitable for stable digital filters. The outward manifestation of this change is in the marked difference in the coefficients of the transfer functions $H_a(s_a)$ and $H(z)$. The view can be taken that the transformation transforms a set of filter coefficients useful for analog realization into another set useful for digital realization. In general, a set of filter coefficients suitable for analog realization is not suitable for digital realization.

As shown in Section 3.1, the transformation given by eq. (122) maps the $j\omega_d$ axis of the s_d plane onto the unit circle in the z plane; every point on the $j\omega_d$ axis maps into a point on the unit circle. However, because of the nature of the exponential function in the transformation eq. (122), every point on the unit circle in the z plane maps into infinitely many points on the $j\omega_d$ axis. This matter is discussed in Sections 3.1 and 3.2, where we show that the entire unit circle maps onto the interval $-\omega_s/2$ to $\omega_s/2$ on the $j\omega_d$ axis, where ω_s is the sampling frequency in radians per second. The mapping then repeats itself periodically on the $j\omega_d$ axis as z moves endlessly around the unit circle. (See Fig. 10.) As the mapping repeats itself periodically on the $j\omega_d$ axis, so $H_d(s_d)$ also repeats itself periodically on the axis.

As stated in item (*iii*) the bilinear transformation in eq. (121) maps every point on the $j\omega_a$ axis of the s_a plane onto the unit circle in the z plane. Thus, points on the $j\omega_a$ axis map through the unit circle to points on the $j\omega_d$ axis in the s_d plane. Under these conditions, the values of the prototype analog transfer function H_a on the $j\omega_a$ axis are repeated as values of the digital transfer function H_d on the $j\omega_d$ axis, as indicated by eq. (118), although in general the values are not distributed in the same way along the two $j\omega$ axes. That is, the frequency characteristics of the digital filter are similar to those of the prototype analog filter, but the frequency scale is warped.

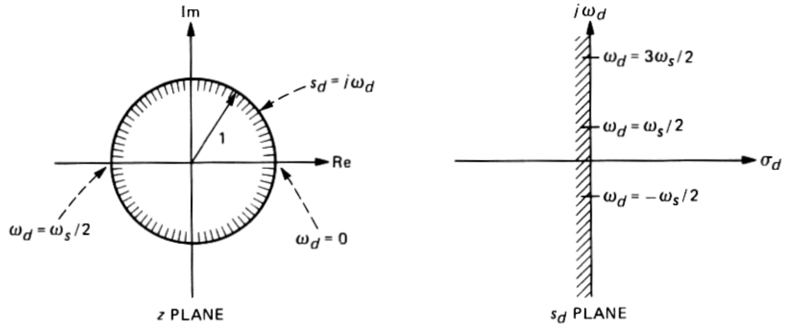


Fig. 10—Relations between the z plane and the s_d plane.

The effectiveness of the bilinear transformation in this application can be highlighted by considering briefly two other transformations that are superficially attractive but that totally fail to produce useful digital filters. The first of these is

$$s_a = \frac{z}{T},$$

where T is the sampling interval. The factor $1/T$ is included as a scale factor and to make the equation dimensionally balanced. This transformation seems attractive because it is so simple. It has the property given in item (*i*) but it does not have the properties of items (*ii*) and (*iii*). Thus, for a given T there are stable analog filters that transform into unstable digital filters. Furthermore, if a stable digital filter is obtained with this transformation, the frequency characteristics of the analog filter are not preserved in the digital filter, and with this transformation it is not possible to obtain a useful digital filter using conventional techniques.

The second transformation to be considered is

$$z = \exp(s_a T)$$

or

$$s_a = \frac{1}{T} \ln z. \qquad (123)$$

Section 3.1 shows that this transformation has the properties listed in items (*ii*) and (*iii*) above. Moreover, the transformation in eq. (123) is the exact inverse of the one given by eq. (122), and applying these two transformations in succession produces $H_d(s_d) = H_a(s_a)$, a digital filter identical with the analog prototype. However, transformation eq. (123) does not have the property listed in item (*i*) above, and hence it cannot be used to obtain a difference equation by the method given in Section 4.4. Without a difference equation, the techniques presented here cannot be used to produce a digital filter. It may be of some interest, but of no real significance, to note that the logarithm in eq. (123) can be represented by an infinite series of the form

$$\ln z = 2\left[\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \cdots\right].$$

Approximating the logarithm by the first term in this series and substituting it into eq. (123) yields

$$s_a = \frac{2}{T}\frac{z-1}{z+1},$$

which is exactly the bilinear transformation of eq. (121).

6.3 Details of the bilinear transformation

The properties of the bilinear transformation presented without proof in Section 6.2 are examined in some detail in this section. The discussion is aided by the diagrams shown in Fig. 11. The transformation is given by eq. (119), and its inverse, given by eq. (120), is

$$z = \frac{1 + s_a T/2}{1 - s_a T/2}. \tag{124}$$

Thus, when $s_a = j\omega_a$,

$$\begin{aligned} z &= \frac{1 + j\omega_a T/2}{1 - j\omega_a T/2} = \frac{Me^{j\phi}}{Me^{-j\phi}} \\ &= e^{j2\phi}, \end{aligned} \tag{125}$$

where

$$\phi = \arctan\,(\omega_a T/2). \tag{126}$$

Thus, when $s_a = j\omega_a$, $|z| = 1$, and hence, the $j\omega_a$ axis in Fig. 11a is mapped by the bilinear transformation onto the unit circle in the z plane as shown in Fig. 11b. Thus, we verify the assertion made to this effect in Section 6.2.

More generally, with $s_a = \sigma_a + j\omega_a$, eq. (124) gives z as

$$z = \frac{1 + \sigma_a T/2 + j\omega_a T/2}{1 - \sigma_a T/2 - j\omega_a T/2} = \frac{N}{D}. \tag{127}$$

Thus,

$$|z| = \left|\frac{N}{D}\right|, \tag{128}$$

with

$$|N| = [(1 + \sigma_a T/2)^2 + (\omega_a T/2)^2]^{1/2}, \tag{129}$$

and

$$|D| = [(1 - \sigma_a T/2)^2 + (\omega_a T/2)^2]^{1/2}. \tag{130}$$

Now, since T is always positive, it follows from eqs. (128), (129), and (130) that

$$|z| < 1 \qquad \text{for all } \sigma_a < 0, \tag{131}$$

and that

$$|z| > 1 \qquad \text{for all } \sigma_a > 0. \tag{132}$$

Thus, the bilinear transformation maps the entire left half of the s_a plane into the interior of the unit circle in the z plane, and it maps the entire right half of the s_a plane into the exterior of the unit circle. Thus, we verify another assertion made in Section 6.2.

After the bilinear transformation has been applied to the prototype analog transfer function $H_a(s_a)$, the transformation given by eq. (122) is used to obtain the digital transfer function $H_d(s_d)$. The results of this transformation are illustrated in Figs. 11b and 11c, and they are

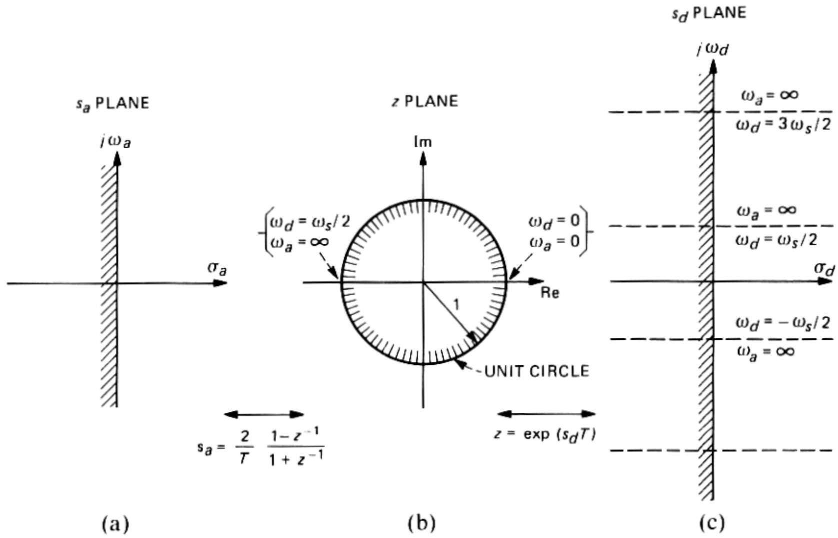


Fig. 11—Transformations and mapping.

discussed in some detail in Section 6.2 in connection with Fig. 10.

Returning to the transformations given by eq. (124),

$$z = \frac{1 + s_a T/2}{1 - s_a T/2}, \tag{133}$$

and eq. (122),

$$z = \exp(s_d T), \tag{134}$$

the following additional information can be deduced. From eq. (133), when

$$s_a = 0, \qquad z = 1, \tag{135}$$

and when

$$s_a = \infty, \qquad z = -1. \tag{136}$$

Similarly, from eq. (134), when

$$s_d = 0, \qquad z = 1, \tag{137}$$

and when

$$s_d = j\omega_s/2, \qquad z = -1, \tag{138}$$

where eq. (138) is worked out in detail in Section 3.1 and ω_s is the sampling frequency in radians per second. These relations are shown in Fig. 11.

As $s_a = j\omega_a$ increases from zero to infinity in Fig. 11a, z moves counterclockwise around the upper half of the unit circle in Fig. 11b from $z = 1$ to $z = -1$. For this movement of z, $s_d = j\omega_d$ increases from zero to $j\omega_s/2$ in Fig. 11c. Thus, as indicated in Fig. 11c, the entire frequency characteristic of the prototype analog filter for all positive values of ω_a is compressed into a segment of the $j\omega_d$ axis between zero and $\omega_s/2$. As indicated by eq. (118), the values of $H_d(j\omega_d)$ in this band vary through exactly the same sequence as the values of $H_a(j\omega_a)$ in the range of ω_a between zero and infinity, but the frequency scale is compressed and distorted. Moreover, the entire frequency characteristic of $H_a(j\omega_a)$ for all frequencies is repeated periodically along the $j\omega_d$ axis, with a warped frequency scale, as indicated in Fig. 11c.

The discussion above shows that when the bilinear transformation is used in the design of a digital filter, there is a close relationship between the frequency characteristics of the prototype analog filter $H_a(j\omega_a)$ and those of the resulting digital filter $H_d(j\omega_d)$. In fact, they are the same except for a warping of the frequency scale. The relationship between the frequency scales can be derived readily from eqs. (125), (126), and (134). Any given value of z lying on the unit circle maps into a point on the $j\omega_a$ axis given by eq. (125) as

$$z = \exp(j2\phi). \tag{139}$$

That same value of z maps into infinitely many points on the $j\omega_d$ axis given by eq. (134) as

$$z = \exp(j\omega_d T \pm 2n\pi), \qquad n = 0, 1, 2, \cdots. \tag{140}$$

Taking the principal value of eq. (140), $n = 0$, and equating it to eq. (139) yields

$$\exp(j\omega_d T) = \exp(j2\phi),$$

or

$$\omega_d T = 2\phi. \tag{141}$$

Substituting eq. (126) into eq. (141) yields

$$\omega_d T = 2 \arctan(\omega_a T/2), \tag{142}$$

and

$$\omega_a = \frac{2}{T} \tan(\omega_d T/2). \tag{143}$$

Using the relation $T = 1/f_s$, where f_s is the sampling frequency, eq. (143) can be rewritten as

$$\omega_a = 2f_s \tan(\omega_d/2f_s).$$

Then,

$$f_a = \frac{f_s}{\pi} \tan\left(\frac{\pi}{f_s} f_d\right) \tag{144}$$

$$= f_d \frac{\tan(\pi f_d/f_s)}{\pi f_d/f_s}. \tag{145}$$

Note that if $f_s \gg f_d$, then

$$f_a \approx f_d.$$

Equation (143), (144), or (145) is used in the design of digital filters when the bilinear transformation is employed. The system designer specifies the desired frequency characteristics in terms of cutoff frequencies, frequencies of attenuation peaks, and the like. These frequencies are critical values of f_d for the digital filter. The critical frequencies are substituted into one of the equations listed above to obtain predistorted critical values of f_a for the prototype analog filter. The analog filter is then designed with these values of f_a, and, because of the predistortion, the digital filter obtained by the bilinear transformation has the desired critical frequencies. Note that every critical frequency contained in the original specifications must be predistorted by one of the equations specified above.

It must also be noted, however, that the bilinear transformation maps the entire frequency characteristic of the prototype analog filter into the range of ω_d lying between zero and $\omega_s/2$. Thus, the designer can control the characteristics of the digital filter only in this range of ω_d, and hence, all of the critical frequencies specified for ω_d must be less than $\omega_s/2$. In this connection, however, it is appropriate to remember that the signal being filtered must be band-limited to this same range of frequencies to avoid distortion as a result of aliasing.

It is clear from eq. (144) that the amount by which the frequency scale is warped depends on the sampling frequency f_s. For a fixed filter design, any change in f_s will cause a shift in the critical frequencies of the digital filter, and if the change is substantial, the result may be an unsatisfactory performance by the filter.

The technique outlined here for the use of the bilinear transformation in the design of digital filters is one possible method that can be used. In addition, several other successful procedures have been developed in detail, and they are described in the literature.[1,3] However, the introductory presentation given here is concerned only with the basic ideas of digital filtering, and hence it makes no attempt to give a complete coverage of all the techniques for designing such filters.

Section 6.2 illustrates that if the digital transfer function $H(z)$ is to be a stable function, then its poles must lie inside the unit circle in the z plane. As we show below, this places a limit on the range of values that the coefficients in $H(z)$ may be permitted to have, and this fact, in turn, may influence the design of the hardware and the supporting software used in realizing digital filters.

As related in Section 4.4, it is a common practice to realize digital filters as a cascade of biquadratic sections used as basic building blocks. The digital transfer function for this building block can be put in the form

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} \tag{146}$$

$$= \frac{a_0 z^2 + a_1 z + a_2}{z^2 + b_1 z + b_2}. \tag{147}$$

Note that since the coefficients in $H(z)$ must be real numbers, the poles and zeros of $H(z)$ must either be real or they must occur in complex conjugate pairs. Furthermore, if the two zeros of the denominator are designated z_1 and z_2, then the denominator can be written as

$$D(z) = (z - z_1)(z - z_2) \tag{148}$$

$$= z^2 - (z_1 + z_2)z + z_1 z_2. \tag{149}$$

Comparing this result with the denominator of eq. (147) leads to the following relations:

$$b_1 = -(z_1 + z_2) \qquad \text{and} \qquad b_2 = z_1 z_2. \tag{150}$$

Since for stable digital filters z_1 and z_2 must lie inside the unit circle in the z plane, the following inequalities must be satisfied:

$$-1 < b_2 < 1 \qquad \text{and} \qquad -2 < b_1 < 2. \tag{151}$$

However, it is appropriate to note the fact that even though the inequalities in eq. (151) are satisfied by the coefficients of $H(z)$, this does not guarantee that the filter is stable. The quadratic formula can be used, with a little additional effort, to show that the necessary and sufficient conditions for stability are

$$-1 < b_2 < 1 \qquad \text{and} \qquad -(1 + b_2) < b_1 < (1 + b_2).$$

See Ref. 6 for further details on this subject.

For most digital filters, the zeros of the numerator in eq. (147) are on the unit circle or inside it. In such cases, after the numerator is normalized to make $a_0 = 1$, the same inequalities apply to the coefficients a_1 and a_2. An exception to this rule occurs in the case of delay equalizers (all-pass networks), where the zeros of the numerator must lie outside the unit circle (in the right half of the s_d plane). But even in this case, it is possible to introduce a scale factor to make the numerator coefficients satisfy inequalities like those in eq. (151).

The significance of these facts arises from the additional fact, developed in Section 4.4, that the coefficients in $H(z)$ are also the coefficients in the difference equation that the digital filter must evaluate in order to perform its function. Thus, it follows that the filter hardware and software can be, and sometimes are, designed to work with coefficients limited to the range between 2 and -2.

Example 4.

This example is a simple illustration of the use of the bilinear transformation to obtain a linear difference equation from a given analog transfer function. It is based on the RC filter of Fig. 1, and the results are compared with those obtained in Section II by a different method of analysis. The analog transfer function for this circuit is

$$H_a(s_a) = \frac{1}{1 + RCs_a}. \tag{152}$$

Substituting the bilinear transformation of eq. (121) into eq. (152) yields

$$H(z) = \left(1 + \frac{2RC}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right)^{-1}. \tag{153}$$

For simplicity, a new symbol is defined as

$$d = \frac{RC}{T} = RCf_s, \tag{154}$$

so that eq. (153) becomes

$$H(z) = \left(1 + 2d\frac{1 - z^{-1}}{1 + z^{-1}}\right)^{-1}.$$

When this equation is rearranged into the standard form, the result is

$$H(z) = \frac{1}{1 + 2d}\;\frac{1 + z^{-1}}{1 + \dfrac{1 - 2d}{1 + 2d}z^{-1}}. \tag{155}$$

This function has a zero at $z = -1$, whereas $H_a(s_a)$ in eq. (152) has no finite zeros. In eq. (155) the zero of $H_a(s_a)$ at infinity has been transformed into a zero of $H(z)$ at $z = -1$, as implied in Fig. 11b by the fact that the point at infinity in the s_a plane maps into the point $z = -1$.

Applying the bilinear transformation to a full biquadratic transfer function involves some tedious algebra, and the tedium is compounded if the required filter consists of several biquadratic sections connected in cascade. Fortunately, computer programs exist that accept the coefficients of the analog transfer function as inputs and deliver the coefficients produced by the bilinear transformation as outputs. (See Ref. 7.)

If the transforms of the input and output voltages in the filter of Fig. 1 are designated, respectively, as $V_1(z)$ and $V_2(z)$, then

$$V_2(z) = H(z)V_1(z), \tag{156}$$

where $H(z)$ is given by eq. (155). To obtain the corresponding difference equation, eq. (155) is substituted into eq. (156), and the result is rearranged to obtain

$$V_2(z) = \frac{1}{1+2d} V_1(z) + \frac{1}{1+2d} z^{-1} V_1(z) - \frac{1-2d}{1+2d} z^{-1} V_2(z). \quad (157)$$

The procedure developed in Section 4.4, eqs. (96) through (101), now yields the desired difference equation,

$$v_2(nT) = \frac{1}{1+2d} v_1(nT) + \frac{1}{1+2d} v_1[(n-1)T] - \frac{1-2d}{1+2d} v_2[(n-1)T]. \quad (158)$$

For comparison, the difference equation obtained by a simpler method in Section II and given by eq. (6) is

$$v_2(nT) = \frac{1}{1+d} v_1(nT) + \frac{d}{1+d} v_2[(n-1)T]. \quad (159)$$

Equations (158) and (159) do not look very much alike. In fact, they do not give similar results except for high sampling frequencies f_s such that $d = RCf_s \gg 1$ and $v_1[(n-1)T] \approx v_1(nT)$. This matter is discussed in the following paragraph.

Curiously enough, the bilinear transformation produces a digital filter having frequency characteristics much closer to those of the prototype analog filter in Fig. 1 than does the seemingly more direct approach used in Section II and represented by eq. (159). The reason for this result can be explained in the following way. The derivation of the difference equation in Section II is equivalent to using the transformation

$$s_a = \frac{1}{T}(1 - z^{-1})$$

in the analog transfer function (eq. 152). Now for the key point: This transformation does not map the $j\omega_a$ axis onto the unit circle in the z plane. Thus, the $j\omega_a$ axis does not map through the z plane onto the $j\omega_d$ axis in the s_d plane, and as a result, the frequency characteristics of the prototype analog filter are not preserved by the transformation. Thus, with this transformation, the digital filter behaves like the analog filter only for high-sampling rates and low-signal frequencies. (See pages 212 through 214 of Ref. 3.)

VII. SUMMARY

In its most usual form the digital filter is a digital machine that performs the filtering process by the numerical evaluation of a linear difference equation in real time under program control.

The z transform, an outgrowth of a special type of Laplace transform, proves to be an especially effective mathematical tool for the analysis and design of digital filters. In fact, the z transform plays a role in the study of linear difference equations that is in many respects comparable to that of the Laplace transform in the study of linear differential equations. In particular, the z transform can be applied to a linear difference equation to obtain an algebraic transfer function. This z-domain transfer function is related to digital filters in the same way that the well-known s-domain transfer function is related to conventional analog filters.

In most cases the design of a digital filter involves determining a digital (z-domain) transfer function having the desired frequency characteristics. This digital transfer function can often be obtained in a straightforward manner from an appropriate analog (s-domain) transfer function. A mathematical relation known as the bilinear transformation proves in many cases to be an especially effective tool for converting an analog transfer function into a useful digital transfer function that can be realized as a digital filter in a straightforward manner.

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