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Minimizing the Worst-Case Distortion in Channel Splitting

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A sequence of outputs from a stationary memoryless source is encoded into n code streams sent over n parallel channels. Any k or fewer of these channels may have broken down, unbeknown to the encoder. The receiver maps the streams from the surviving channels into a reconstruction sequence for minimum distortion. This distortion will take different values depending on what subset of channels is operative. Let D_{max} be the largest of these values, the worst-case distortion. This paper shows that the infimum of D_{max} over all encodings is the same as if the encoder did have knowledge of the breakdown situation.

I. INTRODUCTION

Consider a stationary, memoryless source emitting at each unit of time a random variable X_t with values in a measurable space $\mathscr{X}$. An encoder maps this source stream into n code streams for transmission over n channels going to a common decoder.

The channels have positive capacities

$$C_1 \leq C_2 \leq \cdots \leq C_n, \tag{1}$$

the inequalities following by the choice of the indexing. Up to k of the channels may in fact have broken down, so that

$$K = \sum_{r=0}^{k} \binom{n}{r} \tag{2}$$

situations are possible, but the encoder does not know which of these K situations is realized. The decoder uses the streams from the operative channels to form a sequence of reconstructions $\hat{X}_t$ in a measurable space $\hat{\mathscr{X}}$, (often, but not necessarily, the same as $\mathscr{X}$). Performance is measured by the time average of a distortion function $d(X_t, \hat{X}_t)$. For any given coding scheme, the expected distortion will depend on the breakdown situation. Here we focus on D_{max}, the largest of the K distortions. D_{max} is the expected distortion one can guarantee, subject to the assumption that no more than k of the n channels will break down.

If the k channels with the k highest capacities have broken down, a total capacity

$$\rho = \sum_{i=1}^{n-k} C_i \tag{3}$$

is left. Then even if the encoder knew that this was the situation, the distortion could not be made lower than $\delta(\rho)$, where $\delta(\cdot)$ is the classical distortion-rate function for the given source and distortion measure.[1] A fortiori, one has

$$D_{max} \geq \delta(\rho). \tag{4}$$

In this paper it is shown that this bound is always sharp, i.e.,
Theorem: For $\epsilon > 0$ *it is possible to achieve*

$$D_{max} < \delta(\rho) + \epsilon \tag{5}$$

by using appropriate coding with large enough block length.

Thus, in the problem of minimizing D_{max} one can do as well as if the encoder *did* know which of the K breakdown situations was realized. The price paid for this is that, as will be seen, one has effectively to "throw away" most of the excess over ρ of the capacity available in nonworst situations.

II. REGROUPING OF THE CHANNELS

If the capacities C_i, $i > n - k$ are all reduced to the value C_{n-k} then, by (1) and (3), the value of ρ is unchanged, and if the result holds after such reduction it holds, a fortiori, before the reduction. This means that the extra capacity

$$C_{extra} = \sum_{i=n-k+1}^{n} (C_i - C_{n-k}) \tag{6}$$

can be used for other purposes, such as to reduce the distortion in some situations below D_{max}. Thus, we assume henceforth that $C_i = C_{n-k}$ for $i > n - k$.

The channel coding theorem[1] implies that given $\epsilon > 0$ any channel of capacity C is equivalent—for large enough block length and with appropriate channel coding—to a channel accepting binary bits at rate $C - \epsilon$ and delivering them with arbitrarily small error probability. Thus, we can assume that all n channels are binary with rates $C_i' = C_i$

$-\epsilon_1$ ($i = 1, \cdots, n$), and that they transmit blocks of sufficient size unaltered, with probability $1 - \epsilon_2$, with ϵ_1, ϵ_2 positive, arbitrarily small.

Lemma: For all $\epsilon > 0$, it is possible to transmit sufficiently long blocks of binary bits at rate $\rho - \epsilon$ with error probability less than ϵ, as long as no more than k channels are out of order.

For the proof, let $\gamma_1 = C_1'$ and for $i = 2, \cdots, n-k$,

$$\gamma_i = C_i' - C_{i-1}'. \tag{7}$$

Then one has

$$C_i' = \sum_{j=1}^{i} \gamma_j \tag{8}$$

for $i = 1, \cdots, n-k$, and

$$C_i' = \sum_{j=1}^{n-k} \gamma_j \tag{9}$$

for $i \geq n-k$.

By (1), the $n-k$ numbers γ_i are nonnegative. As a channel of rate

$$\sum_{j=1}^{i} \gamma_j$$

is equivalent to i parallel channels of respective rates $\gamma_1, \gamma_2, \cdots, \gamma_j$, we may consider the following regrouping of these channels:

Group 1 consists of n channels of equal rate γ_1. The ith of these channels is part of the original channel i.

Group 2 consists of $n-1$ channels of equal rate γ_2. They correspond to parts of original channels 2 through n.

Continuing in this fashion:

Group i consists of $n - i + 1$ channels of equal rate γ_i. They correspond to parts of original channels i through n.

Finally, group $n-k$ consists of $k+1$ channels of equal rate γ_{n-k}, corresponding to original channels $n-k$ through n.

Note that for $i = 2, \cdots, n-k$, group i is missing $i-1$ channels corresponding to the first $i-1$ original channels. These missing channels can be viewed as permanently broken down channels of an imaginary group of n. As up to k of the original channels may break down, group i, when considered as originally made up of n channels (of rate γ_i), may have up to $k + i - 1$ broken down channels. This is so for all $n-k$ groups, $i = 1, \cdots, n-k$. Now we invoke the known fact[3] that n channels of equal rate γ_i, out of which at most $k + i - 1$ are out of order, can be used to transmit binary bits error-free at rate $(n - k - i + 1)\gamma_i$ using truncated Reed-Solomon (TRS) codes.[2]

Thus the $n-k$ groups yield a total error-free rate

$$\sum_{i=1}^{n-k} (n-k-i+1)\,\gamma_i = \sum_{i=1}^{n-k} \sum_{j=1}^{i} \gamma_j$$
$$= \sum_{i=1}^{n-k} C_i'$$
$$= \rho - (n-k)\epsilon_1.$$

To split a binary block among the $n - k$ groups and assign to each group an integral number of bits—a multiple of its TRS bloc coding length—rounding may be required with asymptotically negligible losses of rate. In addition, the assumed noiseless behavior of the n channels only holds with probability $(1 - \epsilon_2)^n$. As all the ϵ's involved go to zero as block length increases, the lemma is proved.

Thus, there exist coding schemes, valid in all K situations, which convey data from transmitter to receiver as if a channel of capacity ρ were between them. Then (5) follows from the classical rate-distortion theory.

III. SPECIAL CASES

For a binary symmetric source with Hamming distortion, that is,

$$d(X, \hat{X}) = 0 \text{ when } X = \hat{X},\ 1 \text{ otherwise,}$$

one has

$$\delta(\rho) = h^{-1}(1 - \rho),$$

where $h^{-1}(x) = 0$ for $x \leq 0$, while for $0 < x \leq 1$ it is the inverse of the restriction to (0, ½) of

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x).$$ [1]

For n channels of equal capacity C, of which k can break down, one has $\rho = (n - k)C$ so that the limit of achievability is given by

$$D_{\max} = h^{-1}(1 - (n - k)C). \tag{10}$$

If in particular $C = n^{-1}$ (channels of total capacity 1), then

$$D_{\max} = h^{-1}(k/n). \tag{11}$$

For $k = 1$, if one insists that the distortion approach zero when all n channels are up, one can achieve[4] distortion $(2^{1/n} - 1)/2$ when any channel is down, which is of order n^{-1}. If, however, one only cares about maximum distortion, then one can approach $h^{-1}(1/n)$ which is of order $(n \log n)^{-1}$.

If, for example, $n = 3$ and $k = 1$, then $(2^{1/3} - 1)/2 \simeq 0.130$ while $h^{-1}(⅓) \simeq 0.062$.

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