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Spectral Properties and Band-Limiting Effects of Time-Compressed TV Signals in a Time-Compression Multiplexing System

By K. Y. ENG and O.-C. YUE

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Time-compression multiplexing (TCM) has recently been proposed for application in multiple TV transmissions through satellites. It is advantageous over frequency-division multiplexing because of its relative immunity to nonlinear transponder effects. Here we study two important and fundamental aspects of TCM—the spectral properties and band-limiting effects on the time-compressed signal. We derive the output spectrum of a time-compressed signal and show that if the original input signal is assumed to be: (i) band-limited to B Hz, (ii) segmented into T-second intervals before time compression by a factor of $\alpha(\alpha \geq 1)$, and (iii) $1/T \ll B$, then essentially all the spectral power in the output time-compressed signal is contained in the bandwidth $|f| \leq \alpha B$ Hz. This result is applicable to the TV case. Numerical examples on various types of spectra are also presented. Using the TV example, we further demonstrate that the ripples created by low-pass filtering the time-compressed signal up to αB Hz are small, and interburst interference due to these ripples can be kept negligible with a small guard time (about 2 percent of the burst duration) between different signal bursts. We also provide a brief discussion on some interesting spectral properties of time-compressed signals in spectrum-expansion applications.

I. INTRODUCTION

Time-compression multiplexing (TCM) is a technique whereby multiple signals can be multiplexed together in a common communication channel for transmission.[1,2] A simple illustration of this method is shown in Fig. 1 where $x(t)$ is a continuous waveform intended for transmission. It is first divided into segments of T seconds each; and each segment is time compressed by a factor $\alpha(\alpha \geq 1)$, resulting in a

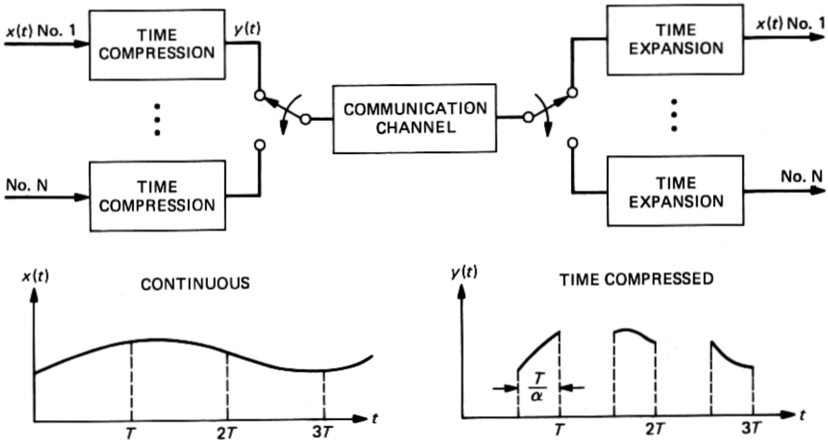

Fig. 1—A simple illustration for TCM.

bursty signal $y(t)$ with a burst duration of T/α seconds. A total of α such time-compressed signals can then be time multiplexed together for transmission. In the particular case of TV transmissions through satellites, TCM is advantageous over frequency-division multiplexing (FDM) because various degrading effects (e.g., intermodulation and intelligible crosstalk) due to transponder nonlinearities can be avoided by employing TCM. In a more general context, TCM is more efficient than FDM whenever time division can be accomplished more efficiently than frequency division. In this paper, we study two important and fundamental aspects of TCM—the spectral properties and the band-limiting effects of the time-compressed signal.

If we assume that the original signal $x(t)$ in Fig. 1 is band-limited to B Hz, then time compressing it by a factor of α in the infinite time duration, i.e., transforming $x(t)$ to $x(t/\alpha)$, would mean a frequency spectrum expansion by the same factor (α). However, as shown in the diagram, $x(t)$ is segmented into T-second intervals before time compression on each segment. Doing so, it is no longer obvious what the spectrum should look like or what the bandwidth expansion factor would be. It is clear though that the spectral power in $y(t)$ is nonzero beyond αB Hz due to the segmentation; and it is desirable that this power beyond αB Hz be small to maintain spectral efficiency in TCM. We derive and discuss an explicit expression for the output spectrum of the time-compressed signal $y(t)$ (see Section II) and show by numerical examples (Section III) that all the significant power is contained in the frequency bandwidth below αB Hz, thus, confirming the long-speculated result that the bandwidth expansion factor in TCM is the same as the time-compression factor.

To ensure compliance with the out-of-band emission requirements, signals are often filtered before transmission. Such a band-limiting operation on the time-compressed signal truncates its small but nonzero power beyond its passband (αB Hz) and creates ripples in its time waveform. The ripples following the trailing edge of each burst are important because they lead to interburst interference in the system. We demonstrate using a computer simulation (Section IV) that in the specific case of TV transmission, (*i*) such a band-limiting effect is minimal as long as all the spectral components below αB Hz are transmitted without distortion and (*ii*) the interburst interference can be kept negligible by introducing a small guard time in the order of two percent of the burst duration between different time-compressed TV signals. These encouraging results on both the bandwidth expansion and band-limiting effects assure us of the basic attractiveness of using TCM to transmit TV signals in nonlinear satellite channels.

It is well-known that time compression can also be used as a means to obtain spectrum expansion, e.g., Henry's spectrum expander used in radio astronomy.[3] In such a case, the key concern is that of spectrum distortion as analyzed thoroughly by Rowe.[4] We extend our results to examine this problem in an appendix, and some simple and interesting spectral properties pertinent to the spectrum expansion application are discussed.

II. SPECTRUM OF THE TIME-COMPRESSED SIGNAL

2.1 Derivation

Referring to Fig. 2, let $x(t)$ be an input signal to an ideal time compressor which performs the time compression on each T-second segment of the input waveform as discussed before. This is mathematically equivalent to first time compressing $x(t)$ in the infinite time duration, resulting in $x_c(t)$, by the required time-compression factor $\alpha(\alpha \geq 1)$, and then time-shifting segments of T/α-second duration in $x_c(t)$ to various proper time instants to arrive at $y(t)$, the desired time-compressed output as shown in the diagram. We are interested in the spectrum (or Fourier transform) of $y(t)$, denoted by $Y(f)$.

By the above definition, $y(t)$ is related to $x_c(t)$ by

$$y(t) = \sum_{k=-\infty}^{\infty} x_c[t - k(T - \tau)]\mathrm{rect}_\tau(t - kT), \tag{1}$$

where

$$\tau \triangleq \frac{T}{\alpha}, \quad (\alpha \geq 1) \tag{2}$$

and

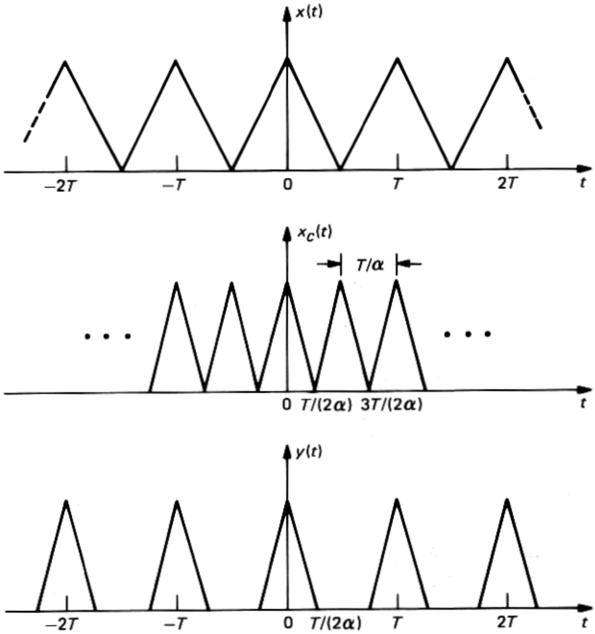

Fig. 2—An illustrative time-compression sequence.

$$\mathrm{rect}_\tau(t) \triangleq \begin{cases} 1, & |t| \le \dfrac{\tau}{2}, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

Using the following,

$$x_c(t) \leftrightarrow X_c(f), \tag{4}$$

$$\mathrm{rect}_\tau(t) \leftrightarrow \tau \ \mathrm{sinc}\ \pi f \tau, \tag{5}$$

where $\leftrightarrow$ denotes Fourier transform pair, and

$$\mathrm{sinc}\ x \triangleq \frac{\sin x}{x}, \tag{6}$$

the Fourier transform of $y(t)$ can be written as

$$Y(f) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X_c(g) \exp[-j2\pi gk(T-\tau)]$$

$$\times \tau \ \mathrm{sinc}[\pi(f-g)\tau] \exp[-j2\pi(f-g)kT]dg. \tag{7}$$

Assuming that the summation and integration can be interchanged, the above becomes

$$Y(f) = \int_{-\infty}^{\infty} X_c(g)\tau \operatorname{sinc}[\pi(f-g)\tau] \sum_{k=-\infty}^{\infty} \times \exp[-j2\pi k(-g\tau + fT)]dg. \quad (8)$$

Applying the well-known identity of

$$\sum_{k=-\infty}^{\infty} \exp(-j2\pi kfT) = \sum_{k=-\infty}^{\infty} \delta(fT-k) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f-\frac{k}{T}\right), \quad (9)$$

$Y(f)$ is simplified as

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} X_c(g)\tau \operatorname{sinc}[\pi(f-g)\tau] \sum_{k=-\infty}^{\infty} \delta(fT-g\tau-k)\,dg \\ &= \sum_{k=-\infty}^{\infty} X_c\left(\frac{fT-k}{\tau}\right) \operatorname{sinc}\left[\pi\left(f-\frac{fT-k}{\tau}\right)\tau\right] \\ &= \sum_{k=-\infty}^{\infty} X_c\left[\frac{T}{\tau}\left(f-\frac{k}{T}\right)\right] \operatorname{sinc} \pi[f(\tau-T)+k]. \end{aligned} \quad (10)$$

Using (2) and

$$x_c(t) = x(\alpha t) \leftrightarrow X_c(f) = \frac{1}{\alpha} X\left(\frac{f}{\alpha}\right), \quad (11)$$

where $X(f)$ is the Fourier transform of $x(t)$, we get the final result of

$$Y(f) = \frac{1}{\alpha}\sum_{k=-\infty}^{\infty} X\left(f-\frac{k}{T}\right) \operatorname{sinc}\left[\pi\left(f-\frac{f}{\alpha}-\frac{k}{T}\right)T\right]. \quad (12)$$

The above expression relates the output spectrum $Y(f)$ to the input spectrum $X(f)$ and is the basic relationship we work with in the rest of the paper. The preceding derivation is the simplest that we are aware of (see Appendix A and Ref. 5 for comparison). We now proceed to discuss some simple properties of $Y(f)$.

2.2 Simple properties

We are interested in the properties of $Y(f)$ that convey information about the spectral occupancy problem (or the bandwidth expansion factor) in TCM systems. In this regard, we must note that $Y(f)$ derived above is just the Fourier transform of one single time-compressed signal in the TCM system. If there are N users for the channel (i.e., N time-compressed signals for transmission), then the total signal in the transmission channel is (without post-time-compression filtering):

$$z(t) = \sum_{i=1}^{N} y_i(t), \quad (13)$$

where each $y_i(t)$ is a time-compressed signal resembling $y(t)$ considered previously. The power spectrum of the total signal in the channel is

$$P(f) = |Z(f)|^2 = \left| \sum_{i=1}^{N} Y_i(f) \right|^2, \tag{14}$$

where $Z(f)$ and $Y_i(f)$ are the Fourier transforms of $z(t)$ and $y_i(t)$, respectively. It is well-known that

$$P(f) = \sum_{i=1}^{N} |Y_i(f)|^2 \tag{15}$$

only when the $y_i(t)$ are all uncorrelated. In the particular case of TV transmission, the various time-compressed TV signals are not totally uncorrelated because of the presence of sync pulses, color subcarrier bursts, and so on. However, from the point of view of spectral occupancy (i.e., the total power contained in some passband), consideration of $Y(f)$ alone is sufficient. In any event, $P(f)$ is calculable from the above if it is needed.

Without loss of generality, we may normalize $T = 1$, and $Y(f)$ becomes

$$Y(f) = \frac{1}{\alpha} \sum_{k=-\infty}^{\infty} X(f-k) \operatorname{sinc}\left[\pi\left(f - \frac{f}{\alpha} - k\right)\right]. \tag{16}$$

The simplest property observable from the above is the output dc component in $Y(f)$, i.e.,

$$\begin{aligned} Y(0) &= \frac{1}{\alpha} \sum_{k=-\infty}^{\infty} X(-k) \operatorname{sinc}(\pi k) \\ &= \frac{1}{\alpha} X(0), \end{aligned} \tag{17}$$

which holds for any general $X(f)$ (see Ref. 5 for comparison).

Let us now examine the bandwidth property of $y(t)$. We assume that the input signal is band-limited to B Hz, i.e., $X(f)$ is zero for $|f| > B$. With the normalization of $T = 1$, $X(f)$ is band-limited to $M = B/T$. Dropping the multiplying constant of $1/\alpha$ for convenience, and at a particular frequency $f = \alpha f_x$ (recall that α is the time-compression factor),

$$Y(\alpha f_x) = \sum_{k=-\infty}^{\infty} X(\alpha f_x - k) \operatorname{sinc} \pi[f_x(\alpha - 1) - k], \tag{18}$$

where the sinc function provides weightings for various points in $X(f)$. Since the sinc function is maximum when its argument is zero, it is sensible to perform the summation starting with the value of k that maximizes the sinc function. Denoting such a value of k by k_0, it is given by solving

$$f_x(\alpha - 1) - k_0 = 0. \tag{19}$$

The solution is

$$k_0 = f_x(\alpha - 1) + \epsilon, \quad |\epsilon| \leq 0.5, \tag{20}$$

where ϵ is necessary because k_0 is restricted to be integer, and k_0 is unique, except for the case $\epsilon = \pm 0.5$. It should be emphasized that ϵ depends on both f_x and α. Using this expression for k_0, $Y(\alpha f_x)$ can be written as

$$\begin{aligned} Y(\alpha f_x) &= \{X(\alpha f_x - k) \operatorname{sinc} \pi[f_x(\alpha - 1) - k]\}_{k=k_0} \\ &\quad + \sum_{k \neq k_0} X(\alpha f_x - k) \operatorname{sinc} \pi[f_x(\alpha - 1) - k] \\ &= X(f_x - \epsilon) \operatorname{sinc} \pi(-\epsilon) \\ &\quad + \sum_{K \neq 0} X(f_x - \epsilon + K) \operatorname{sinc} \pi(-\epsilon + K), \end{aligned} \tag{21}$$

where K in the summation is taken as $K = \pm 1, \pm 2$, and so on. A graphical representation of the above is depicted in Fig. 3.

To get immediate insight into the bandwidth property, we consider the following two cases:

Case 1: α = integer, f_x = integer. Under this assumption, $\epsilon = 0$ and we have a simple relationship of

$$Y(\alpha f_x) = X(f_x). \tag{22}$$

This means that every integer value of f in $X(f)$ is mapped exactly onto αf in $Y(f)$. Without the normalization on T, this is equivalent to saying that every integer multiple of $1/T$ in $X(f)$ is mapped exactly onto α/T in $Y(f)$ as shown in Fig. 4. If the condition that $1/T \ll B$ holds, it is almost certain that the spectrum $Y(f)$ is simply the

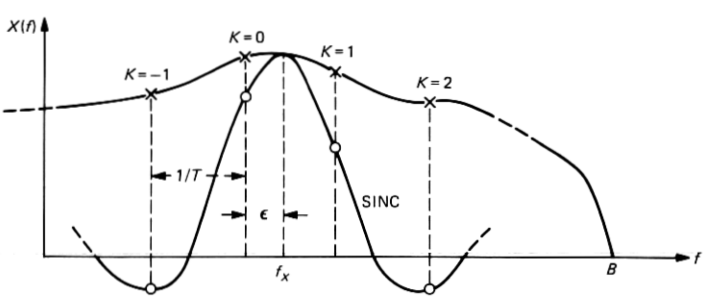

Fig. 3—Graphical illustration for the summation in the expression for the output spectrum of a time-compressed signal.

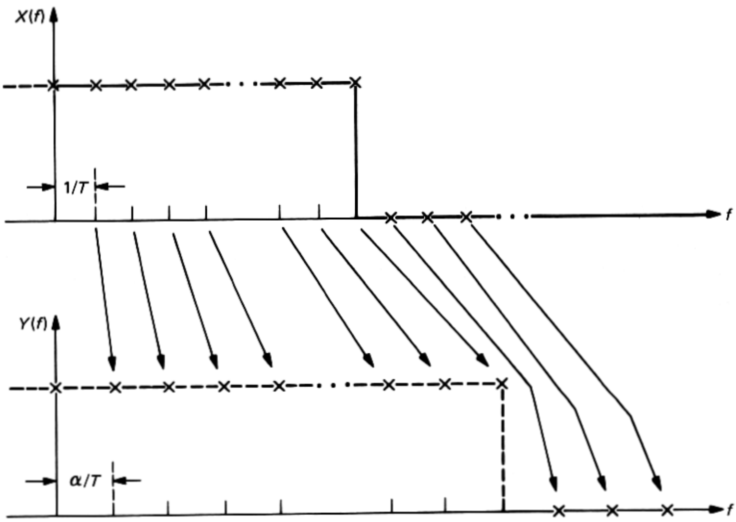

Fig. 4—Frequency-spectrum expansion property through time compression.

frequency-expanded version of $X(f)$ by the factor α with some insignificant sidebands for $|f| > \alpha B$ Hz. In the particular case of TCM/TV, T is taken as a scan line duration, yielding $1/T \approx 15.73$ kHz. With $B =$ 4.2 MHz, $BT \approx 267$, using α as the bandwidth expansion factor is, therefore, a good rule of thumb for the TV case. We note that the condition of α and f_x being integers is merely an artifact due to normalization. Therefore, we emphasize that this case is indeed more general than it appears to be. For instance, for any given set of α, we can always find a set of f_x such that this condition holds.

Case 2: $\alpha \neq$ integer, $f_x \neq$ integer. ϵ is generally nonzero here, and $Y(\alpha f_x)$ is given by a weighted sum of $X(f_x - \epsilon + k)$ [see (21) and Fig. 3] with $X(f_x - \epsilon)$ as the main contributor. An alternative view is that $Y(\alpha f_x)$ is the weighted average of $X(f_x - \epsilon)$ and its neighboring points. The output spectrum $Y(f)$ is again a frequency-expanded version of $X(f)$, except for ripples created by the averaging process. It is noted that ϵ can be zero here resulting in $Y(\alpha f_x) = X(f_x)$. This occurs whenever $f_x(\alpha - 1)$ is an integer in (20), e.g., $\alpha = 3.5$, $f_x = 106.8$.

The foregoing discussion by and large answers the basic question on bandwidth expansion in TCM systems. The result of using the time-compression factor α as the bandwidth-expansion factor makes sense for most cases where the input spectrum $X(f)$ can be modeled as continuous and band-limited. The inclusion of peculiar nulls and delta functions in $X(f)$ would complicate the matter, but it can be examined in detail using the equations provided above. As to the precise shape of $Y(f)$ in comparison to $X(\alpha f)$, which is relevant in the spectrum expansion application, some interesting discussions are given in Appendix B.

III. NUMERICAL EXAMPLES OF OUTPUT SPECTRUM

We present in this section specific numerical examples of output spectra calculated from (12). There are four different types of $X(f)$ in the examples:

(*i*) Rectangular

$$X(f) = \begin{cases} 1, & |f| \le B, \\ 0, & \text{otherwise}. \end{cases} \tag{23}$$

B is normalized to be 1 Hz in the calculation, and T is taken as $267/B$ seconds (the TV case). Results for five different values of the time-compression factor α are plotted in Fig. 5. The peak-to-peak ripples in the output passband ($|f| \le B$) are less than 0.5 dB and the sidebands are more than 25 dB down in the vicinity beyond the edge of the passband and drop off very rapidly. There is no doubt that most of the spectral power is contained in the bandwidth $|f| \le B$.

(*ii*) Triangular

$$X(f) = \begin{cases} 1 - \dfrac{|f|}{B}, & |f| \le B, \\ 0 \quad , & \text{otherwise}. \end{cases} \tag{24}$$

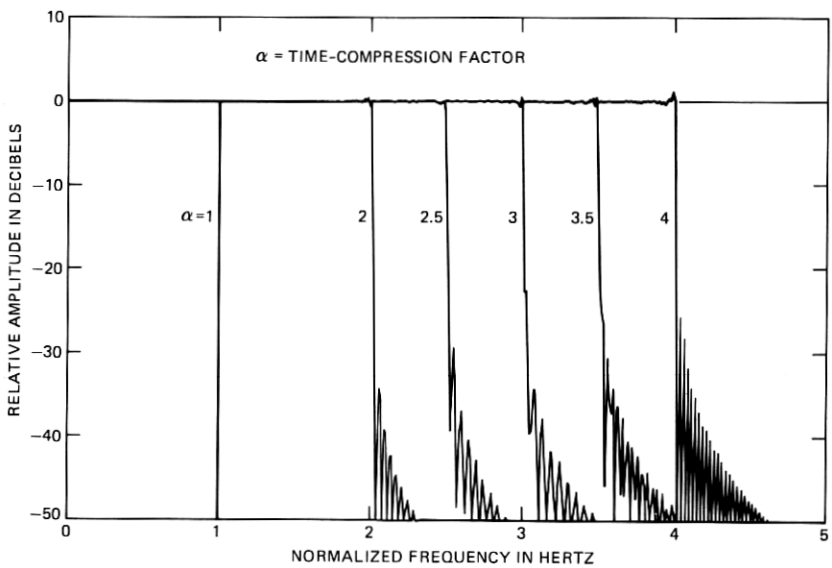

Fig. 5—Output spectrum of time-compressed signal with a rectangular input spectrum.

B is normalized to be 1 Hz, and $T = 267/B$ seconds. The results are plotted in Fig. 6. Here the sidebands are so low that they are not observable in the diagram. Of course, this is due to the taper-off characteristic in $X(f)$. Some small ripples are again present inside the bandwidth $|f| \le \alpha B$.

(*iii*) Half-Cosine

$$X(f) = \begin{cases} \cos\left(\dfrac{\pi}{2}\dfrac{|f|}{B}\right), & |f| \le B, \\ 0 & , \text{otherwise.} \end{cases} \tag{25}$$

B is again 1 Hz and $T = 267/B$ seconds. The results are plotted in Fig. 7, and the same observations as in (*ii*) apply here.

(*iv*) Truncated Half-Cosine

The equation for $X(f)$ is the same as in (*iii*) above, except that B is 0.9362 Hz, which corresponds to a 20-dB taper at $X(B)$ as compared to $X(0)$. T is again taken as 267 seconds. The results are shown in Fig. 8 where the glitches at the edge of the output passband (i.e., $f \approx \alpha B$) are evident. Note that the sidebands outside the passband are much lower compared to those in (*i*) above.

IV. BAND-LIMITING EFFECTS

When the original input signal $x(t)$ is band-limited to B Hz, we have shown that most of the power in the output time-compressed signal

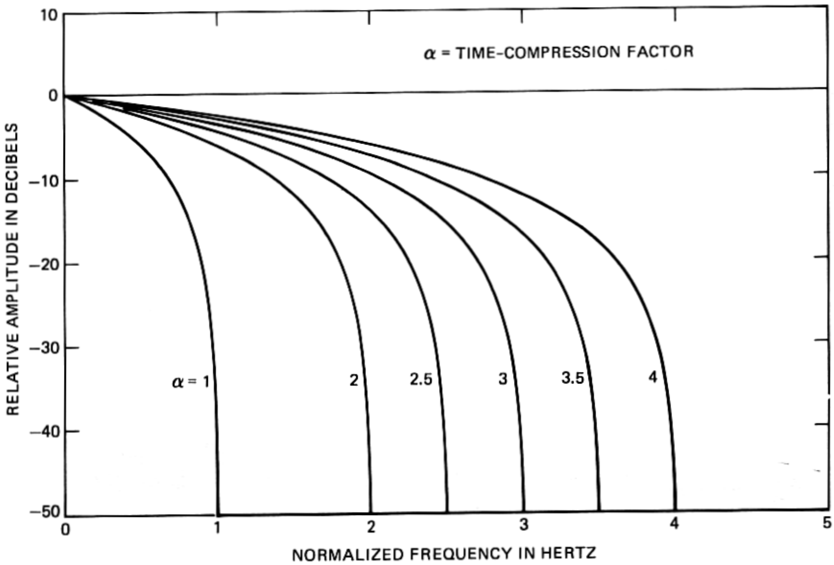

Fig. 6—Output spectrum of time-compressed signal with a triangular input spectrum.

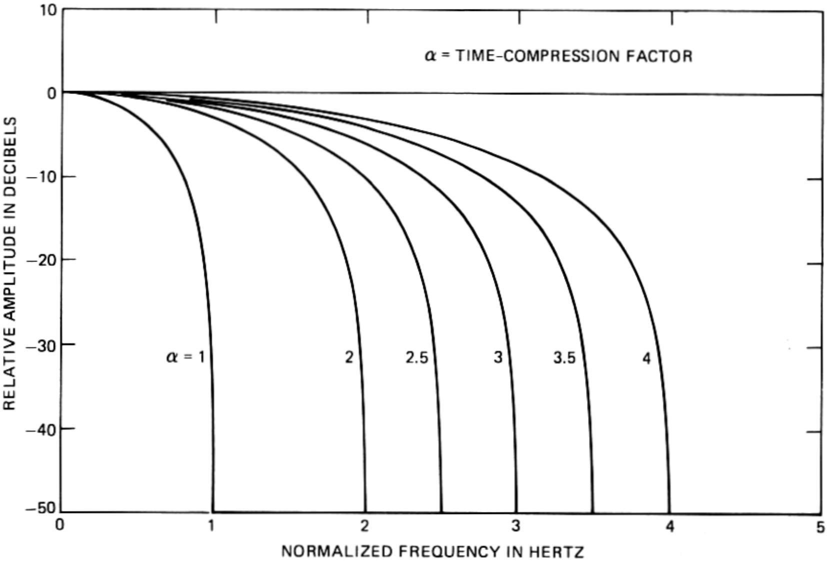

Fig. 7—Output spectrum of time-compressed signal with a half-cosine input spectrum.

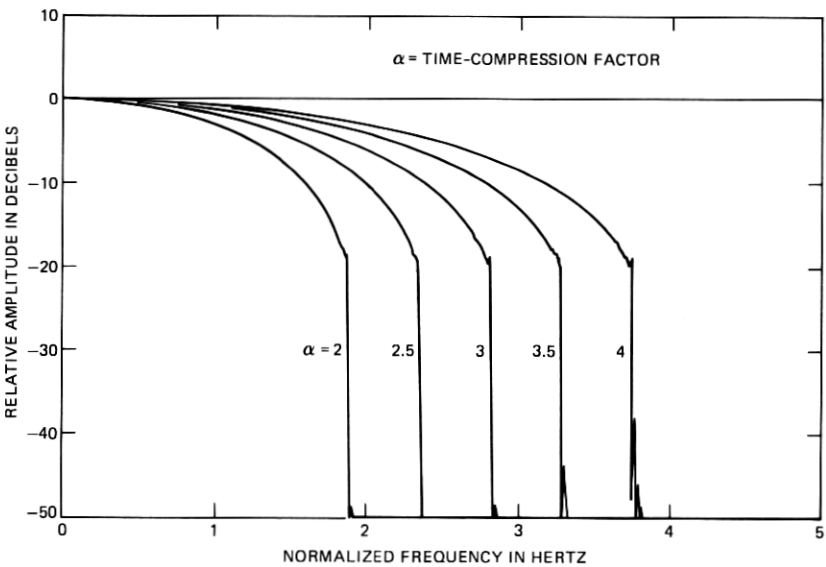

Fig. 8—Output spectrum of time-compressed signal with a 20-dB tapered half-cosine spectrum.

$y(t)$ is confined to $|f| \leq \alpha B$ Hz. Band-limiting $y(t)$ up to αB Hz therefore should hardly affect the fidelity of the signal itself. However, such filtering does create ripples following each time-compressed signal burst, where a sharp edge occurs in the time waveform. These ripples constitute interburst interference and could be a potential source of degradation in a TCM system. We demonstrate via a simulation on a TV signal in this section that the problem can be alleviated by introducing a small guard time (about 2 percent of the burst duration) between time-compressed signal bursts from different users.

In our computer simulation, we generate a TV test signal which is similar to the composite test signal in Ref. 6. A scan line (64 μs) of this is shown in Fig. 9. We first band-limit the test signal by a low-pass (LP) filter with zero delay and a raised-cosine amplitude roll-off of

$$H(f) = \begin{cases} 1 & , |f| \leq F_1 + \dfrac{1-r}{2t_0}, \\ \dfrac{1}{2}\left\{1 - \sin\left[\dfrac{t_0\pi(|f| - F_1)}{r} - \dfrac{\pi}{2r}\right]\right\} & , \dfrac{1-r}{2t_0} \leq |f| - F_1 \leq \dfrac{1+r}{2t_0}, \\ 0 & , \text{otherwise.} \end{cases} \tag{26}$$

where the parameters F_1, t_0 and r ($F_1 \geq 0$, $t_0 \geq 0$; $0 \leq r \leq 1$) control the

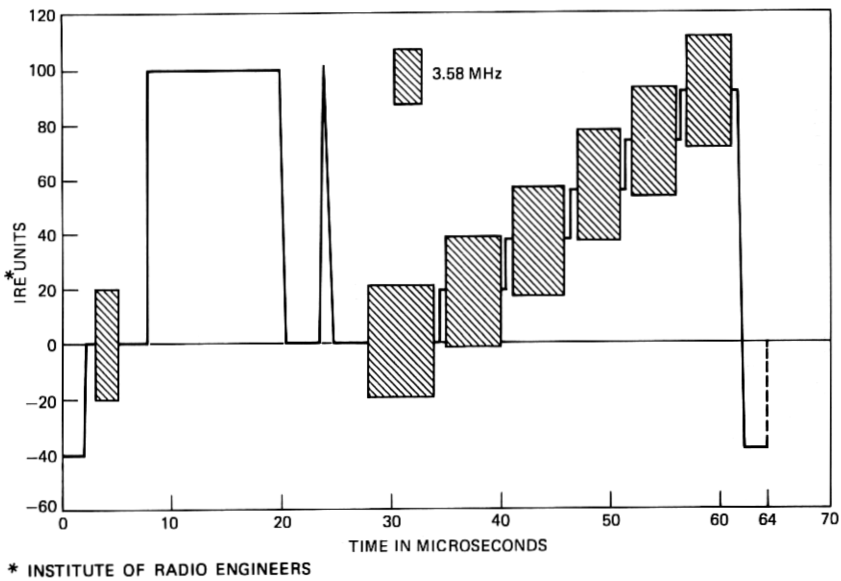

Fig. 9—A modified composite test signal.

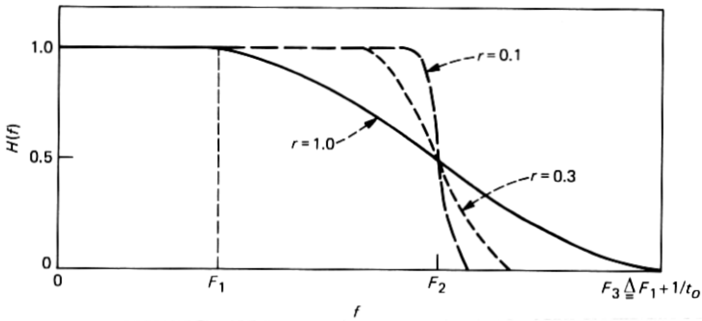

Fig. 10—A low-pass filter with a raised-cosine roll-off.

filter shape. Instead of using t_0 and r to define the filter shape, we use the two critical frequencies defined by (Fig. 10):

$$F_2 \triangleq F_1 + \frac{1}{2t_0}; \tag{27}$$

$$F_3 \triangleq F_1 + \frac{1+r}{2t_0}. \tag{28}$$

The specific values for F_1, F_2, and F_3 are 4.2 MHz, 4.8 MHz, and 5 MHz, respectively.

After the initial band-limiting, we perform an ideal time compression over each scan line, and the signal is then compressed (with $\alpha = 2$) into time bursts, each of which is 32 μs long. We then filter the time-compressed signal by the LP filter referred to above with F_1, F_2, and F_3 at 8.4 MHz, 9.6 MHz, and 10 MHz, respectively. The ripples following each 32-μs signal burst are observed, and the data are plotted in Fig. 11 where the ripple magnitude is defined as the ratio of the peak-to-peak ripple voltage to the peak-to-peak picture voltage. As seen from the figure, a guard time of 0.5 μs is sufficient for controlling the interburst interference. This amounts to about 2 percent of the burst duration.

V. CONCLUSION

We have studied two important and fundamental aspects of TCM—the spectral properties and band-limiting effects of the time-compressed signal. We find under the assumptions that (*i*) the original input signal $x(t)$ is band-limited to B Hz, (*ii*) $x(t)$ is segmented into T-second intervals before time compression by a factor of $\alpha (\alpha \geq 1)$, and (*iii*) $1/T \ll B$, then essentially all the spectral power in the output time-compressed waveform is confined to frequencies below αB Hz. This result is immediately applicable to the TV case. Numerical ex-

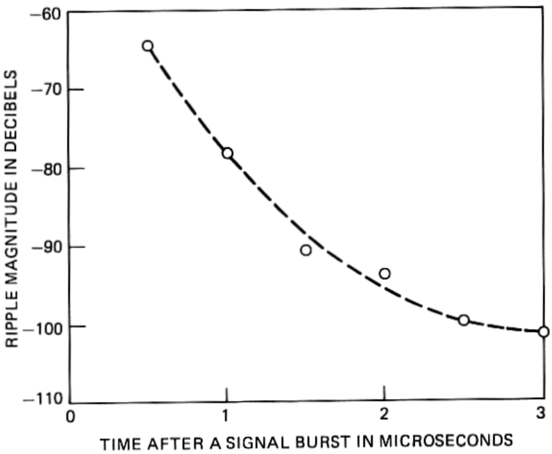

Fig. 11—Ripple magnitude following a time-compressed signal burst in the TV simulation.

amples on various types of spectra verify this and show that the spectral sidebands beyond αB Hz are very low. We also find that filtering of a time-compressed TV signal creates small ripples following each signal burst, but the interburst interference due to these ripples can be kept negligible with a small guard time (about 2 percent of the burst duration) between different signal bursts.

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APPENDIX A

An Alternate Derivation for the Spectrum of the Time-Compressed Signal

Referring to Fig. 2, the input signal can be written as

$$x(t) = \sum_{i=-\infty}^{\infty} x(t)\ \mathrm{rect}_T(t - iT), \tag{29}$$

where $\mathrm{rect}_T(t)$ is defined in (3). The output time-compressed signal is

$$y(t) = \sum_{i=-\infty}^{\infty} x\left\{\alpha\left[t - iT\left(1 - \frac{1}{\alpha}\right)\right]\right\} \mathrm{rect}_\tau(t - iT), \tag{30}$$

where α is the time-compression factor, and τ is as defined in (2). Note that

$$\mathrm{rect}_\tau(t - iT) \leftrightarrow \tau \,\mathrm{sinc}(\pi f \tau) \exp(-j2\pi f i T), \tag{31}$$

$$x\left\{\alpha\left[t - iT\left(1 - \frac{1}{\alpha}\right)\right]\right\} \leftrightarrow \frac{X\left(\frac{f}{\alpha}\right)}{\alpha} \exp\left[-j2\pi f i T\left(1 - \frac{1}{\alpha}\right)\right]. \tag{32}$$

The spectrum of $y(t)$ involves the convolution of the two right sides in (31) and (32) and is

$$Y(f) = \tau \sum_i \int_{-\infty}^{\infty} X(\beta) \exp[-j\beta i T(\alpha - 1)]$$

$$\times \mathrm{sinc}[\pi\tau(f - \beta\alpha)] \exp[-j(2\pi f - \beta\alpha) i T] d\beta. \tag{33}$$

Using the identity of (9), the above can be rewritten as

$$Y(f) = \tau \sum_i \int_{-\infty}^{\infty} X(\beta) \,\mathrm{sinc}[\pi\tau(f - \beta\alpha)] \delta\left[T\left(f - \frac{\beta}{2\pi}\right) - i\right] d\beta. \tag{34}$$

With a change of variable of

$$\sigma = T\left(f - \frac{\beta}{2\pi}\right), \tag{35}$$

$Y(f)$ becomes

$$Y(f) = \frac{1}{\alpha} \sum_i \int_{-\infty}^{\infty} X\left(\frac{Tf - \sigma}{T}\right) \mathrm{sinc}\left[\pi\tau\left(f - \frac{Tf - \sigma}{T}\alpha\right)\right] \delta(\sigma - i) d\sigma$$

$$= \frac{1}{\alpha} \sum_i X\left(f - \frac{i}{T}\right) \mathrm{sinc}\left[\pi\tau\left(f - f\alpha + \frac{i\alpha}{T}\right)\right]$$

$$= \frac{1}{\alpha} \sum_i X\left(f - \frac{i}{T}\right) \mathrm{sinc}\left[\pi\left(f - \frac{f}{\alpha} - \frac{i}{T}\right)T\right], \tag{36}$$

which is the same as (12).

A simple way to check $Y(f)$ is of course to let $\alpha = 1$ in (36), which yields

$$Y(f) = \sum_i X\left(f - \frac{i}{T}\right) \mathrm{sinc}\, \pi(-i)$$

$$= X(f), \tag{37}$$

as expected. Another way to check $Y(f)$ involves letting $\alpha \to \infty$. In doing so, we have to assume that the energy in $x_c(t)$ is preserved through the time compression, i.e.,

$$\int_{-\infty}^{\infty} x_c^2(t)\,dt = \int_{-\infty}^{\infty} x^2(t)\,dt. \tag{38}$$

This means that the multiplying factor $1/\alpha$ in (36) can be dropped, and with $\alpha \to \infty$,

$$Y(f) = \sum_i X\left(f - \frac{i}{T}\right) \text{sinc}\left[\pi\left(f - \frac{i}{T}\right)T\right]. \tag{39}$$

We now try to verify (39) by a different means. Using $\alpha \to \infty$ as described above, the output time waveform is

$$y(t) = \sum_i \delta(t - iT) \int_{iT - \frac{T}{2}}^{iT + \frac{T}{2}} x(\tau)\,d\tau$$

$$= \sum_i \delta(t - iT)\bar{y}(iT). \tag{40}$$

Note that if $\bar{Y}(f) \leftrightarrow \bar{y}(t)$, then

$$\frac{1}{T}\sum_k \bar{Y}\left(f - \frac{k}{T}\right) \leftrightarrow \sum_i \delta(t - iT)\bar{y}(iT). \tag{41}$$

Consider now a general definition of

$$\bar{y}(t) \triangleq \int_{t - \frac{T}{2}}^{t + \frac{T}{2}} x(\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\,\text{rect}_T(t - \tau)\,d\tau. \tag{42}$$

The Fourier transform of $\bar{y}(t)$ is then

$$\bar{Y}(f) = X(f)[T\,\text{sinc}(\pi f T)]. \tag{43}$$

Using the above, (40) and (41), we get the desired result of (39).

APPENDIX B

Additional Spectral Properties of the Time-Compressed Signal

With $T = 1$, we show in (21) that the spectral output of $f = \alpha f_x$ in $Y(f)$ is

$$Y(\alpha f_x) = X(f_x - \epsilon)\,\text{sinc}\,\pi(-\epsilon)$$

$$+ \sum_{K \neq 0} X(f_x - \epsilon + K)\,\text{sinc}\,\pi(-\epsilon + K), \tag{44}$$

where ϵ depends on both α and f_x. Whenever $\epsilon = 0$, we get

$$Y(\alpha f_x) = X(f_x), \tag{45}$$

which is a desirable result for spectrum-expansion applications. Consider now a simple example, where $\alpha = 2.1$ and $X(f)$ consists of two spectral lines at $f = \pm 5$ Hz (a constant sine wave). We expect two spectral lines in $Y(f)$ at $f = \pm 2.1 \times 5 = \pm 10.5$ Hz. But using $f_x = 5$ in (44), we have

$$Y(10.5) = X(5 - 0.5) \operatorname{sinc} \pi(0.5) + \sum_{K \neq 0} X(5 - 0.5 + K) \operatorname{sinc} \pi(-0.5 + K) = 0, \tag{46}$$

which is so because each term containing $X(f)$ in the summation is identically zero as $X(f)$ admits nonzero values only at $f = \pm 5$ Hz. This absence of output at $f = 10.5$ is due to the fact that $Y(f)$ admits nonzero outputs only at $f =$ integer, or equivalently in multiple spacings of $1/T$ in the unnormalized case. The segmentation of the input signal $x(t)$ into T-second intervals can therefore be viewed as making the frequency resolution in $Y(f)$ $1/T$.

As mentioned previously, when $\epsilon \neq 0$ in (44), the output $Y(\alpha f_x)$ can be viewed as some weighted average of $X(f_x - \epsilon)$ and its neighboring points. How close is this average to the value $X(f_x)$? We do not have a satisfactory answer, but the following discussion is interesting. Let us change the notation in (44) to

$$\hat{X}(f_x) = X(f_x - \epsilon) \operatorname{sinc} \pi(-\epsilon) + \sum_{K \neq 0} X(f_x - \epsilon + K) \operatorname{sinc} \pi(-\epsilon + K), \tag{47}$$

where $\hat{X}(f_x)$ denotes an interpolated or estimated value for $X(f)$. A physical interpretation on the above was presented in Fig. 3. An alternate view is to rewrite the second term in the above as

$$\sum_{K \neq 0} X(f_x - \epsilon + K) \operatorname{sinc} \pi[(f_x - \epsilon + K) - f_x], \tag{48}$$

which is graphically interpreted in Fig. 12. We now do the following manipulations:

$$\begin{aligned}\hat{X}(f_x) &= \sum_K X(f_x - \epsilon + K) \operatorname{sinc} \pi[(f_x - \epsilon + K) - f_x] \\ &= \sum_K X(f_x - \epsilon + K) \int_{-1/2}^{1/2} \exp\{j2\pi t[(f_x - \epsilon + K) - f_x]\}\, dt \\ &= \int_{-1/2}^{1/2} \sum_K X(f_x - \epsilon + K) \exp[j2\pi t(f_x - \epsilon + K)] \\ &\quad \times \exp(-j2\pi f_x t)\, dt.\end{aligned} \tag{49}$$

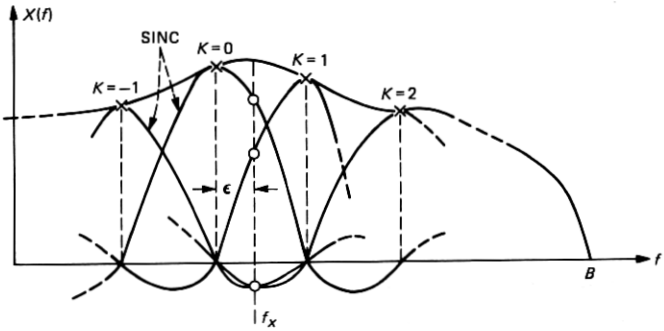

Fig. 12—Alternative representation for the summation in the output spectrum of a time-compressed signal.

We note that the term

$$\sum_K X(f_x - \epsilon + K) \exp[j2\pi t(f_x - \epsilon + K)] \tag{50}$$

is a Fourier series with a period of unity. Denoting this Fourier series by $\hat{x}(t)$, we see that

$$\begin{aligned} \hat{x}(t) &= \exp[j2\pi t(f_x - \epsilon)] \sum_K X(f_x - \epsilon + K) \exp(j2\pi Kt) \\ &= \sum_k x(t-k) \exp[j2\pi k(f_x - \epsilon)], \end{aligned} \tag{51}$$

which is a complicated summation of various time- and phase-shifted versions of $x(t)$.

We now extend our previous results to a limited case applicable to Refs. 5, 6. Consider the special case where α is an integer, denoted by N. We take the time-compressed waveform $y(t)$ in Fig. 2, time shift it by iT/N ($i = 0$ to $N - 1$) and add all N waveforms together. The resultant waveform has no blank time interval and is basically a staggering of N time compressed $y(t)$ which we have considered so far. This can be written as

$$z(t) = \sum_{i=0}^{N-1} y\left(t - \frac{iT}{N}\right), \quad N \geq 2. \tag{52}$$

Its Fourier transform is

$$\begin{aligned} Z(f) &= Y(f) \sum_{i=0}^{N-1} \exp\left(-j2\pi f \frac{iT}{N}\right) \\ &= Y(f) \frac{1 - \exp(-j2\pi fT)}{1 - \exp\left(-j2\pi f \dfrac{T}{N}\right)}, \end{aligned} \tag{53}$$

where $Y(f)$ is the Fourier transform of $y(t)$. The magnitude of the second complex factor is

$$\left|\frac{1-\exp(-j2\pi fT)}{1-\exp\left(-j2\pi f\frac{T}{N}\right)}\right| = \sqrt{\frac{1-\cos 2\pi fT}{1-\cos\left(2\pi f\frac{T}{N}\right)}}. \tag{54}$$

The above term vanishes whenever only the numerator vanishes, and is nonzero when both numerator and denominator vanish simultaneously. This means that the above term is zero when

$$f=\frac{k}{T}, \quad (k = \text{integer}), \tag{55}$$

except at those points where

$$f=k\frac{N}{T} \tag{56}$$

holds. Therefore, one may infer that the frequency resolution in this case is N/T as compared to $1/T$ in the single $y(t)$ case.