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The Limit of the Blocking As Offered Load Decreases With Fixed Peakedness

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For a simple overflow process (SOP) with intensity a *and fixed peakedness* z *offered to a blocking system with* N *trunks, it is shown that the blocking approaches* $(1-1/z)^N$ *as* a *approaches zero. Further, for* $N > 1$ *and* a *sufficiently small, it is shown that the blocking increases as* a *decreases, while for* $N = 1$ *the blocking decreases monotonically. This result is helpful in restricting the range of use of algorithms based on simple overflow processes; for* $N > 1$ *when blocking less than the limit is encountered, nonmonotone behavior of the blocking function can be expected as the offered load is reduced. The present result is closely related to one of A. A. Fredericks, who found that the peakedness of the overflow from* N *trunks offered an SOP with intensity* a *and fixed peakedness* z *approaches* z *as* a *approaches zero. To give an intuitive explanation of Fredericks' result and the present one, it is shown that both would follow if the SOP asymptotically behaves as a thin batch process in which the batch size has a geometric distribution.*

I. INTRODUCTION

A blocking system is a trunk group with no queueing; when an arriving call does not find an idle trunk immediately, it overflows from the system. Every trunk group mentioned here is assumed to be a blocking system and to have independent exponential service times with unit mean. Also, statistical equilibrium is always assumed. If the arrival stream to a trunk group is Poisson, the stream of overflow calls is termed a simple overflow process (SOP). An SOP is characterized by two parameters; generally, these are taken to be the intensity and peakedness, denoted here as a and z, respectively. (We follow Fredericks[1] for terminology and notation as well as for proofs or references not given here.)

A trunk group with SOP input is the model underlying the Equivalent Random method[2] of sizing trunk groups that carry traffic whose peakedness is greater than unity. Algorithms based on the Equivalent Random method, or on approximations to it, are used to relate trunk-group size to the parameters of the input traffic to achieve a given probability of blocking (blocking). When these algorithms iterate on a, with z (and possibly other parameters) fixed, to achieve a low blocking on a given trunk group, they have been found sometimes not to converge, most prominently for large values of z.[3] This will certainly happen if there exists a minimum blocking, over all values of a, for the given values of the other parameters, and the target blocking is less than this minimum. An algorithm may fail also if it requires that the blocking be a monotone function of a when relevant values of a are in a region where this function is nonmonotone. If the blocking has a positive limit as a approaches zero and also has a unique minimum, one can avoid algorithmic failure owing to these causes by avoiding blocking values less than the limit, or at least treating them with caution. In what follows we find the limit of the blocking as a decreases to zero, which is positive for z greater than unity; and, although we do not prove the uniqueness of the minimum, we show that the limit is approached from below for all trunk groups with more than one trunk.

II. THE LIMIT OF THE BLOCKING

When an SOP is offered to a group of N trunks, the blocking on this group is denoted as $B(N, a, z)$. When the offered stream is Poisson, the blocking is written $B(N, a)$, given by the well-known formula

$$B(N, a) = e^{-a}a^N \Big/ \int_a^\infty e^{-x}x^N dx,$$

$$= (a^N/N!) \Big/ \sum_{i=0}^{N} a^i/i!$$

for N integral.

The following is true:

$$\text{For } z \text{ fixed,} \qquad \lim_{a\to 0} B(N, a, z) = (1 - 1/z)^N. \tag{1}$$

To prove (1) we note that there exist a Poisson load A and a (nonintegral) number of trunks T such that

$$a = AB(T, A)$$

$$z = A/(T + 1 + a - A) + 1 - a, \tag{2}$$

in which case

$$B(N, a, z) = B(T + N, A)/B(T, A). \tag{3}$$

Because the case of N integral is particularly simple, we treat it separately. Since $B(T, A)$ is continuous (in both arguments), and as a decreases to zero, T increases monotonically without limit (for fixed $z > 1$), we can find a decreasing sequence of values of a for which T is integral; and thus for the present we treat T as an integer without further comment.

From (3) and using the formula for $B(T, A)$, after a little simplification,

$$B(N, a, z) = A^N \Big/ \left\{ (T+1)^{(N)} \left[1 + a \sum_{i=1}^{N} A^{i-1}/(T+1)^{(i)} \right] \right\}, \quad (4)$$

where $T^{(i)} = T(T+1) \cdots (T+i-1)$.

For the case $N = 1$, we show that $B(N, a, z)$ strictly decreases to its limit $1 - 1/z$ as $a \to 0$. From (4),

$$\begin{aligned} B(1, a, z) &= A/\{(T+1)[1 + a/(T+1)]\} \\ &= A/(T+1+a). \end{aligned} \quad (5)$$

From (2), we obtain

$$A/(T+1+a) = 1 - 1/(z+a). \quad (6)$$

From (6) it is clear that $B(1, a, z)$ strictly decreases to $1 - 1/z$ as $a \to 0$. Furthermore, we note from (6) that

$$A/(T+1) \to 1 - 1/z. \quad (7)$$

From (4), using (7),

$$B(N, a, z) \sim [A/(T+1)]^N \to (1 - 1/z)^N. \quad (8)$$

III. ASYMPTOTIC BEHAVIOR OF THE BLOCKING

In this section, the requirement that N be integral is relaxed, and we investigate the magnitude of $B(N, a, z)$ in the neighborhood of its limit.

For real N we write

$$\begin{aligned} B(N, a, z) &= \frac{A^N T!}{(T+N)!} \cdot \frac{\dfrac{1}{T!}\displaystyle\int_A^\infty e^{-x} x^T dx}{\dfrac{1}{(T+N)!}\displaystyle\int_A^\infty e^{-x} x^{T+N} dx} \\ &= \frac{A^N T!}{(T+N)!} \cdot \frac{Q[2A|2(T+1)]}{Q[2A|2(T+N+1)]}, \end{aligned} \quad (9)$$

where $Q = Q(\chi^2|\nu)$ is the complementary χ^2 distribution function (d.f.) with ν degrees of freedom.

Since $A/(T+1) \to 1 - 1/z$ for $a \to 0$, both χ^2 d.f.'s will approach unity. An estimate of the rapidity of this approach will be needed below. First,

$$1 - Q[2A|2(T+1)] \sim P(x),$$

where

$$x = (A - T - 1)/\sqrt{T+1}$$

and $P(x)$ is the standard normal d.f.[4] Also, $P(x) = Q(-x)$, where $Q(x)$ is the complementary normal d.f., and it is known[5] that

$$\begin{aligned} Q(x) &\sim 1/(x\sqrt{2\pi})\exp(-x^2/2) \\ &= o\{1/(\sqrt{2\pi})\exp[-(T+1-A)^2/2(T+1)]\} = o(T^{-k}) \end{aligned}$$

for any finite k. Since we will be interested in expanding $\log B(N, a, z)$ only as far as infinitesimals of order T^{-1}, the χ^2 d.f.'s in (9) can be ignored. If we use Stirling's formula and (7),

$$\begin{aligned} B(N, a, z) &\sim A^N T^{T+1/2} e^{-T}/[(T+N)^{T+N+1/2} e^{-(T+N)}] \\ &\sim (A/T)^N/[(1+N/T)^{T+N+1/2} e^{-N}] \\ &\sim (1-1/z)^N (1+1/T)^N/[(1+N/T)^{T+N+1/2} e^{-N}]. \quad (10) \end{aligned}$$

Also,

$$\begin{aligned} \log B(N, a, z) &\sim \log(1-1/z)^N + N\log(1+1/T) \\ &\qquad - (T+N+½)\log(1+N/T) + N \\ &\sim \log(1-1/z)^N - (N^2 - N)/2T. \quad (11) \end{aligned}$$

Therefore, for $N > 1$, $B(N, a, z)$ increases to its limit, $(1-1/z)^N$, for sufficiently small a. This indicates that the dip in the load-blocking curves for fixed z shown in Ref. 6 (pages 30 to 32) exists for every $N > 1$.

IV. AN ASYMPTOTIC STRUCTURAL CONJECTURE

To provide some intuitive understanding of his result that the peakedness of the overflow from N trunks approaches z as $a \to 0$, Fredericks says, "That is, by fixing z and letting $a \to 0$ we are charging the 'structure' of the process to one that is 'infinitely bunchy'." In an attempt to extend the provision of intuitive understanding of Fredericks' result and the present result, a conjectural structure is heuristically developed below for an SOP with a infinitesimal and z fixed.

Overflows from the equivalent group of T trunks, which constitute the SOP, occur only when that group is busy. The mean length of a busy period is $1/T$ and the mean number of overflowing calls during a busy period is $A/T \to 1 - 1/z$. With the mean time between overflows becoming infinite, the mean time between busy periods, or clusters of busy periods, must also become infinite. Thus, the SOP will consist of clusters of calls occurring during infinitesimal intervals (random numbers of closely spaced busy periods) separated by intervals that become infinitely large. The limiting structure appears to be that of a "thin" batch process.

Since we are concerned with the peakedness, we assume that the SOP is offered to an infinite trunk group and define the random variable X as the number of busy trunks at an arbitrary instant on such a group. We define a thin process as any call arrival process with an infinitesimal parameter λ, for which the generating function of the distribution of X may be written

$$G(s) = 1 - \lambda \int_0^\infty \{1 - \beta(e^{-t}s + 1 - e^{-t})\}dt + o(\lambda), \tag{12}$$

in which β is a probability generating function on the positive integers. (The generalization to other service-time d.f.'s is apparent.) The factorial-moment generating function corresponding to $G(s)$ is

$$F(s) = 1 - \lambda \int_0^\infty \{1 - \beta(e^{-t}s + 1)\}dt + o(\lambda), \tag{13}$$

and what might be called the factorial-cumulant generating function is

$$\log F(s) = -\lambda \int_0^\infty \{1 - \beta(e^{-t}s + 1)\}dt + o(\lambda). \tag{14}$$

Since the factorial cumulants bear the same relation to the factorial moments that the ordinary cumulants do to the ordinary moments, a comparison of (13) and (14) shows that for a thin process the cumulants are the same as the corresponding moments up to infinitesimals of higher order.

The mean of X is

$$m = \lambda b_1 + o(\lambda), \tag{15}$$

where b_1 is the mean batch size. The second factorial moment of X, from (13) or (14), is $\lambda b_2/2 + o(\lambda)$, where b_2 is the second factorial moment of the batch size. Hence, the peakedness of a thin SOP is

$$z = b_2/2b_1 + 1 + o(1). \tag{16}$$

If the probability of a batch of size x is pq^{x-1}, $q = 1 - p$, $x = 1$,

. . . , then the probability that the batch size is $N + x$, given that it is at least $N + 1$, is

$$pq^{x+N-1} \Big/ \sum_{i=0}^{\infty} pq^{N+i} = pq^{x-1} \tag{17}$$

(the "lack of memory" of the geometric distribution). Since a thin input process results in a thin overflow process and since the peakedness of a thin process depends only on the batch-size distribution, Fredericks' result would follow if the SOP asymptotically is a thin geometric-batch process.

The first two factorial moments of a geometric distribution with no mass at zero are $b_1 = 1/p$, $b_2 = 2q/p^2$. Hence,

$$p = 1/z,$$

where, as in the following, infinitesimals are dropped. Next, we find the blocking met by a thin geometric-batch process offered to N trunks. Since asymptotically all the blocking derives from intra-batch interference, this is given by

$$B = p \sum_{i=1}^{\infty} q^{N+i-1} \cdot \frac{N+i}{1/p} \cdot \frac{i}{N+i}, \tag{18}$$

in which the last factor on the right-hand side is the conditional blocking probability, given that the call arrives in a batch of size $N + i$. The other factors together give the probability of such an arrival. (For a discussion see Ref. 7.)

Hence,

$$B = p^2 q^N \sum_{i=1}^{\infty} iq^{i-1}$$

$$= q^N = (1 - 1/z)^N, \tag{19}$$

which is known to be correct.

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