THE BELL SYSTEM TECHNICAL JOURNAL

DEVOTED TO THE SCIENTIFIC AND ENGINEERING ASPECTS OF ELECTRICAL COMMUNICATION

Volume 61 | February 1982 | Number 2

Effects of Day-to-Day Load Variation on Trunk Group Blocking

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(Manuscript received September 4, 1981)

Modern trunking theory recognizes the need to account for day-to-day load variation when sizing a trunk group for an average blocking objective. This paper investigates the effects of high levels of load variation on average blocking, the measure of service used for sizing final trunk groups in the Public Switched Network. Specifically, we identify a curious phenomenon in which high day-to-day variation results in low average blocking and characterize the traffic theoretic models for which this occurs. By a similar analysis, we also investigate the behavior of an alternate measure of service, the probability of blocking, measured by the ratio of the number of unsuccessful attempts to the total number of attempts.

I. INTRODUCTION

1.1 Background

An important class of trunk groups in the Public Switched Network consists of those groups that provide the last-choice route for a call trying to reach its destination. The performance of such a "final trunk group" is defined to be the 20-day average blocking during the chosen busy hour of the busy season. In the Bell System, the final groups are sized for an objective of one percent average blocking ($\bar{B}$.01) in the busy season.

Figure 1 shows the history of the measured busy-hour loads offered to a trunk group over consecutive days. The degree of variability of the load measurements is not consistent with the hypothesis that the

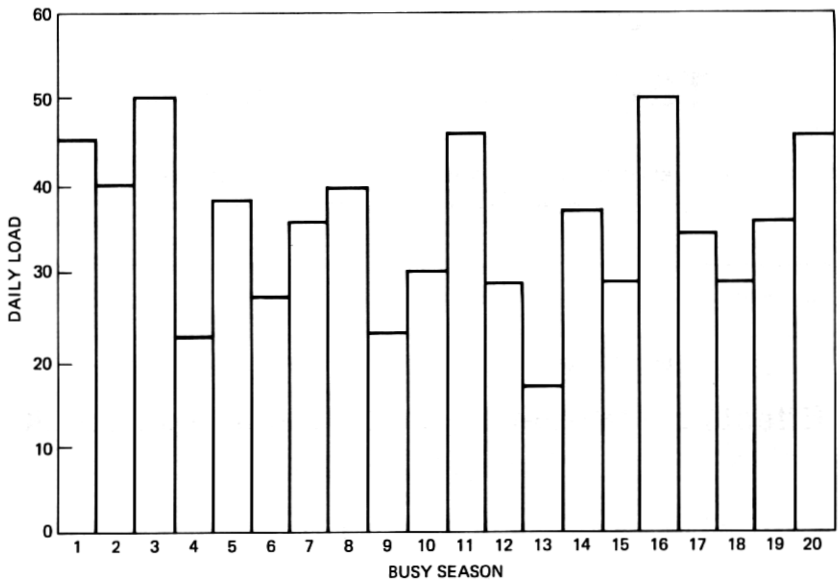

Fig. 1—Busy-season load variations.

daily busy-hour load is constant; rather, it varies from day-to-day in a somewhat random fashion. Modern teletraffic theory has recognized that this day-to-day variability in the offered load must be considered when determining the number of trunks required for an objective level of average blocking. It is generally believed that this variability tends to increase a group's average blocking, and therefore, its trunk requirement, over that observed during constant load conditions.

1.2 Motivation

Recent analyses of Bell Operating Company traffic data have revealed that levels of day-to-day load variation, much higher than those considered in current trunk engineering practices, occasionally appear in the network. To study the effects of such traffic on network service and trunk requirements, the currently deployed traffic models were extended into this region of high load variability. The results are illustrated in Fig. 2, which shows, for Poisson traffic with a fixed mean offered load, $\bar{a}$, and number of trunks, c, the average blocking $\bar{B}$ as a function of the variance, v, of the daily offered load. This quantity, called the day-to-day load variation, is usually parameterized by the variable ϕ, with $v = 0.13\bar{a}^{\phi}$ [1, 2]. In current engineering practices, ϕ is assumed to vary between 1.0 and 1.84. The graph shows that $\bar{B}$ increases with increasing load variation, as expected, but only up to a certain point. Beyond that point, $\bar{B}$ decreases monotonically to zero.

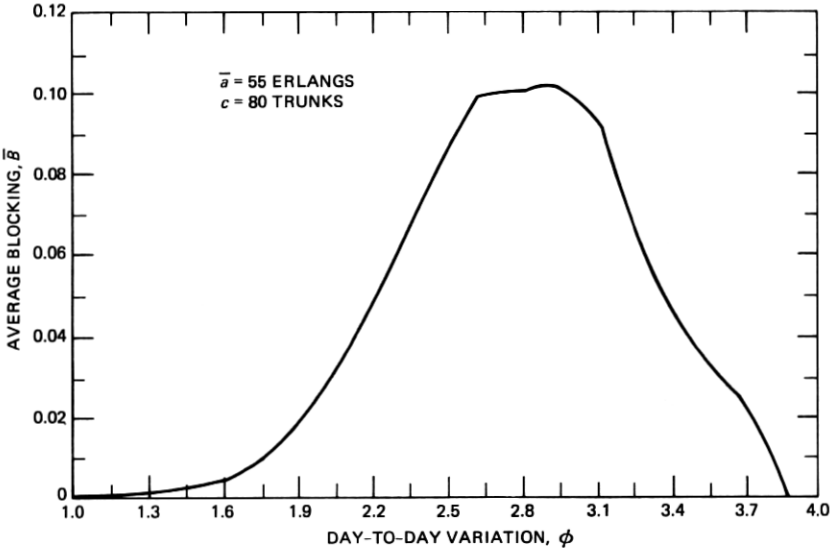

Fig. 2—Average blocking versus day-to-day variation.

Thus, the model predicts that a call arrival process with a highly variable daily load will have lower average blocking than one with a constant daily load.

The purpose of this paper is to analyze and explain this counter-intuitive phenomenon and to determine its implications on current service objectives and traffic models.

1.3 Overview

Section II reviews the model of day-to-day load variation currently deployed in Bell System trunk engineering practices. We then define the average blocking service criterion used for sizing final trunk groups and compare it to an alternate measure of service, the probability of blocking. In Section III, we analyze the behavior of the two measures of service under conditions of high day-to-day load variability and compare numerically their properties in different regions of engineering interest. Section IV summarizes our results and discusses the implications of our findings on trunk engineering practices and network service objectives.

II. TRAFFIC MODELS

This section reviews the model of day-to-day load variation that, in conjunction with models describing the within-hour call arrival process, is used to predict the relationship between trunk group size, average load, and average blocking.

2.1 Average blocking

The discovery of the effect of day-to-day load variation on average blocking is credited to a 1958 study by Wilkinson,[3] who proposed a mathematical model[4] that is now the basis for many of the Bell System's trunk engineering practices.

In Wilkinson's model, the load, a, offered to a group of c trunks within a time-consistent hour (e.g., 9 am to 10 am) varies from day to day in a random fashion. Specifically, the daily loads are modeled as independent random variables with a common gamma distribution $\Gamma(a\,|\,\bar{a}, v)$ with mean $\bar{a}$ and variance v. We assume further that within each hour, the call arrival process is Poisson and that blocked calls do not retry. The daily blocking probability for a trunk group with c trunks and offered load a is defined by the Erlang blocking function, $B(c, a)$; the average daily blocking probability, denoted $\bar{B}$, is given by

$$\bar{B} = \bar{B}(c, \bar{a}, v) = \int_0^\infty B(c, a)d\Gamma(a\,|\,\bar{a}, v). \tag{1}$$

2.2 Other considerations

Wilkinson's model (1) has been used within the Bell System to account for day-to-day load variation in the sizing of final trunk groups. Current trunk engineering practices also account for (i) the non-Poisson or "peaked" nature of overflow traffic,[5] and (ii) the finiteness of the one-hour measurement interval within which traffic measurements are collected.[1] However, the consideration of these additional factors affects only the choice of the daily blocking function in (1) but not the results of this paper, which hold for any single hour blocking function of practical interest.

2.3 Probability of blocking

In the next section, we will compare certain properties of the average blocking service measure, $\bar{B}$, with those of another standard service measure, the probability of blocking, defined below.

Let

$$\mathscr{o}(c, a) = aB(c, a) \tag{2}$$

denote the overflow from a trunk group with c trunks and offered load a, and let

$$\bar{\mathscr{o}} = \int_0^\infty aB(c, a)d\Gamma(a\,|\,\bar{a}, v) \tag{3}$$

denote the average overflow. We define the probability of blocking, P_B, as the ratio of the average overflow to the average offered load:

$$P_B = \frac{\bar{o}}{\bar{a}}$$

$$= \frac{1}{\bar{a}} \int_0^\infty aB(c, a) d\Gamma(a \mid \bar{a}, v). \quad (4)$$

Thus, P_B is simply the probability that an arriving call is blocked.

Comparing eqs. (1) and (4), we note that $\bar{B}$ is an unweighted average of the daily blockings, whereas P_B is a weighted average in which the daily blocking is weighted by the daily load.[4]

Next, we investigate the behavior of both $\bar{B}$ and P_B under conditions of high-load variability.

III. AVERAGE VERSUS PROBABILITY OF BLOCKING

In this section, we develop analytical results that validate and explain the high-variation, low-blocking phenomenon observed in Section I. We begin by relating the asymptotic behavior of $\bar{B}$ to the properties of the assumed daily load distribution.

3.1 Asymptotic behavior of $\bar{B}$

First, let us generalize the definition of average blocking given in (1) to the case in which the daily loads are independent, nonnegative random variables with a common distribution function $F(a)$. To simplify notation, let $B(a)$ denote the daily blocking probability on a particular trunk group expressed as a function of its offered load a. Then, the unweighted average blocking is the quantity

$$\bar{B} = \int_0^\infty B(a) dF(a). \quad (5)$$

Clearly, (5) coincides with Wilkinson's model (1) when $B(a)$ is the Erlang B blocking probability and the daily loads are gamma-distributed. By defining the average blocking in the more general form, we can investigate the role of both the blocking function $B(a)$ and the load distribution $F(a)$ in determining the behavior of $\bar{B}$.

Let $\bar{a}$ and v denote the mean and variance of the offered load distribution. Our first goal is to find general conditions on $F(a)$ under which $\bar{B} \rightarrow 0$ as $\sqrt{v}/\bar{a}$, the coefficient of variation of $F(a)$, increases. Our second goal is to show that, for a very general class of blocking functions $B(a)$, the commonly assumed gamma distribution satisfies the required conditions on the load distribution. Thus, by analysis we will both verify and explain the high-variation, low-blocking behavior described in Section I.

To give a precise answer to the first question posed above, we first define the general class of blocking functions to be considered. A realvalued function $B(a)$ will be called a blocking function if it has the following four properties:

($B1$) $B(0) = 0$ and $B(a) > 0$ if $a > 0$;

($B2$) B is continuous;

($B3$) B is non-decreasing; and

($B4$) B is bounded.

Let $\{F_k\}$ be a sequence of probability distribution functions concentrated on $[0, \infty)$ and consider the sequence

$$\bar{B}_k = \int_0^\infty B(a)dF_k(a). \tag{6}$$

In Appendix A, we derive simple, necessary and sufficient conditions on the distribution sequence $\{F_k\}$ under which $\bar{B}_k \to 0$. Specifically, this will occur if and only if

$$\lim_{k\to\infty} F_k(a) = 1 \quad \text{for all} \quad a > 0. \tag{7}$$

The sufficiency of (7) follows from a standard convergence theorem (See Ref. 6, p. 249). A simple, direct proof of both the necessity and sufficiency of (7) is given in Appendix A.

Condition (7) says that all quantiles of the load distribution converge to zero. Equivalently, this means that all of the mass of the distribution converges toward the origin. Note, however, that (7) does not imply that the moments of the distributions (e.g., mean, variance) converge to zero.

Using this result, we can now analyze the effect of high day-to-day load variation on average blocking in the case of gamma-distributed daily loads.

Let $\{\Gamma_k(a)\}$ denote a sequence of gamma distribution functions with mean $\bar{a}_k$ and variance v_k. According to our result, the average blocking $\bar{B}_k = \int_0^\infty B(a)d\Gamma_k(a)$ converges to zero if and only if

$$\lim_{k\to\infty} \Gamma_k(a) = 1$$

for all $a > 0$. In Appendix B, we show that this occurs when the coefficient of variation of $\Gamma_k(a)$, $\sqrt{v_k}/\bar{a}_k$, increases without bound. In particular, if $\bar{a}_k = \bar{a}$ is fixed and $v_k \to \infty$, the average blocking $\bar{B}_k \to 0$ for arbitrary $c > 0$. Moreover, the number of trunks required to guarantee one percent blocking, $\bar{B}.01$, also tends to zero if $\bar{a}_k = \bar{a}$ and $v_k \to \infty$.

Thus, the results of this section give theoretical explanation for the phenomenon observed in Section I.

3.2 Asymptotic behavior of P_B

The results of the previous section can also be used to analyze the asymptotic behavior of P_B, the probability of blocking, which we defined in Section II.

Recall that the probability of blocking can be expressed as

$$P_B = \frac{\bar{o}}{\bar{a}} = \frac{1}{\bar{a}} \int_0^{\infty} o(a) dF(a), \tag{8}$$

where $o(a)$ is the daily overflow load and $F(a)$ is the distribution of the daily offered load.

Let $u(a) = a - o(a)$ denote the daily carried load. Then

$$\bar{o} = \bar{a} - \bar{u}, \tag{9}$$

where $\bar{u} = \int_0^\infty u(a)dF(a)$ is the average carried load. Thus,

$$P_B = \frac{\bar{a} - \bar{u}}{\bar{a}}. \tag{10}$$

We can now analyze the effect of day-to-day load variation on the probability of blocking by studying the behavior of $\bar{u}$. To do this, we first note that the function $u(a)$ has the following properties:

($u1$) $u(0) = 0$ and $u(a) > 0$ for $a > 0$;
($u2$) $u(a)$ is continuous;
($u3$) $u(a)$ is non-decreasing; and
($u4$) $u(a)$ is bounded (by the number of trunks in the group).

Thus, the carried load $u(a)$ has the required properties of the "blocking function" and our result of Section 3.1 can be applied. That is, we know that $\bar{u} \to 0$ if and only if the load distributions converge as in (7). If $\bar{u} \to 0$ and $\bar{a}$ is fixed, then

$$P_B = \frac{\bar{a} - \bar{u}}{\bar{a}} \to 1. \tag{11}$$

In particular, if the daily loads are gamma-distributed, the probability of blocking converges to 1 as the coefficient of day-to-day variation increases with the mean held constant.

These results illustrate dramatically a fundamental difference between the average blocking ($\bar{B}$) and probability of blocking (P_B) service criteria under conditions of highly volatile network loads. For extremely high levels of day-to-day load variation, the average blocking measure takes on low values, indicating good service, even though most of the traffic is blocked ($P_B \approx 1$). Under such conditions, the unweighted average blocking is a poor indicator of the service experienced by the customer.

3.3 Comparison of $\bar{B}$ and P_B

In addition to the asymptotic results described above, we can also analyze the general relationship between $\bar{B}$ and P_B. Specifically, we will show that $P_B \geq \bar{B}$ regardless of the load distribution.

To see this, note that our assumption that the blocking $B(a)$ is a nondecreasing function of the offered load implies that for any load distribution $F(a)$:

$$\int_0^{\bar{a}} (\bar{a} - a)B(a)dF(a) \leq \int_0^{\bar{a}} (\bar{a} - a)B(\bar{a})dF(a)$$

$$= \int_{\bar{a}}^{\infty} (a - \bar{a})B(\bar{a})dF(a) \leq \int_{\bar{a}}^{\infty} (a - \bar{a})B(a)dF(a). \tag{12}$$

Equivalently,

$$\int_0^{\infty} aB(a)dF(a) \geq \bar{a} \int_0^{\infty} B(a)dF(a),$$

or

$$P_B = \frac{\bar{o}}{\bar{a}} \geq \bar{B}. \tag{13}$$

Next, we complement these results with a few numerical examples that illustrate the differences between $\bar{B}$ and P_B.

3.4 Numerical examples

Recall that final trunk groups in the public telephone network are sized for an average blocking objective of one percent. The behavior of $\bar{B}$ in this low blocking region is illustrated by the lower curve in Fig. 3, which shows, as a function of the day-to-day load variation, the actual average blocking experienced by a trunk group sized under the assumption of no day-to-day variation. As we observed in Section I, $\bar{B}$ initially increases to a maximum value much less than 1.0, and then decreases to zero. In contrast, the probability of a call's being blocked, P_B, monotonically increases to 1.0. However, within the range of day-to-day variation normally encountered ($1.0 \leq \phi \leq 1.84$), the numerical difference between $\bar{B}$ and P_B is not great.

Finally, let us examine the behavior of $\bar{B}$ on a trunk group sized for a high level of average blocking (e.g., 20 percent), assuming no day-to-day variation. Fig. 4 illustrates that day-to-day variation exerts an altogether different influence in this region. Namely, the average blocking decreases monotonically to zero, although the probability of a call's being blocked again increases monotonically to 1.0.

IV. SUMMARY AND CONCLUSIONS

This paper describes an investigation of certain properties of the

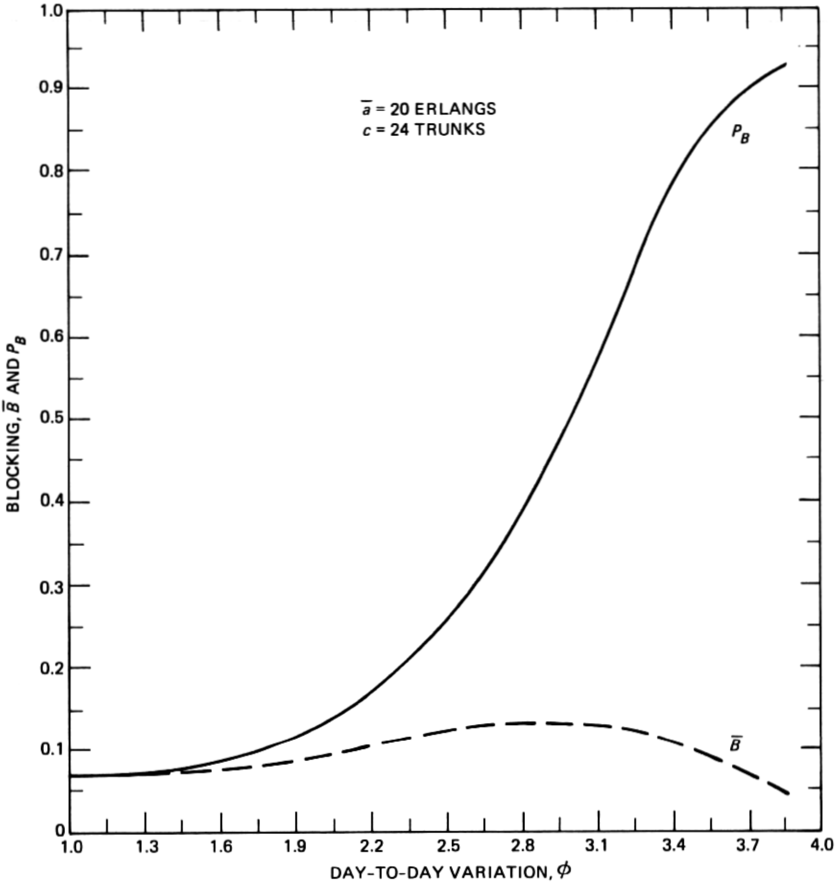

Fig. 3—Blocking versus day-to-day variation.

average blocking service measure ($\bar{B}$) used for sizing final trunk groups in the Public Switched Network. The work was motivated by the results of a recent data study that suggested that levels of day-to-day load variation much higher than those considered by Wilkinson[4] and current Bell System engineering practices occasionally appear in the network. The analytical development presented in Section III confirmed our numerical result of Section 1.2 that very high levels of load variation will result in low levels of average blocking. In addition, the properties of an alternate measure of blocking, P_B, were analyzed and compared to those of $\bar{B}$.

Our findings are summarized below:

(*i*) For the low, objective level of average blocking (one percent) and the traffic conditions normally encountered in the Public Switched

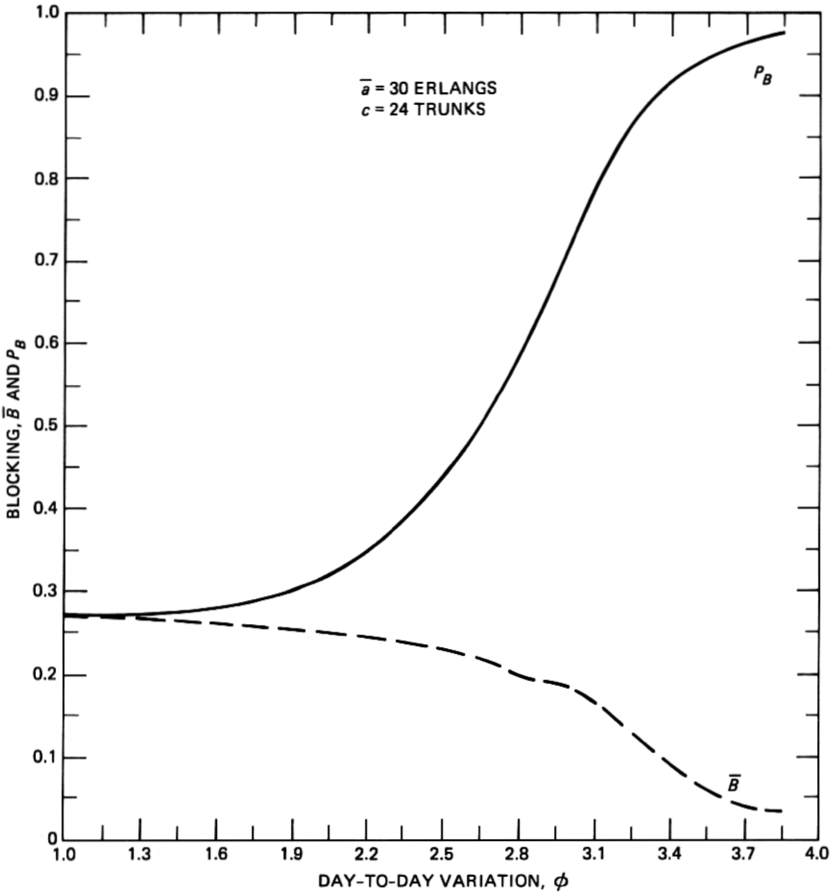

Fig. 4—Blocking versus day-to-day variation.

Network, the discrepancy between $\bar{B}$ and P_B is not great. That is, the current Bell System practice of sizing final trunk groups for a fixed, low level of average blocking also yields correspondingly low, though not uniform, levels of blocking probability.

(*ii*) In contrast with $\bar{B}$, engineering for a fixed level of P_B guarantees, on average, a fixed fraction of successful calls. However, in the presence of high day-to-day variation, the number of trunks required for an objective level of P_B is substantially greater than that required for the same level of $\bar{B}$. Because, the choice of an engineering objective should consider both customer satisfaction and cost, we cannot conclude that P_B is a better objective than $\bar{B}$.

(*iii*) In the range of high blocking, which is typical for private and/or special services networks, the discrepancy between $\bar{B}$ and P_B is substantial. In that case, our results suggest that consideration should be given to appropriateness of average blocking as a service measure.

(*iv*) We showed that, under volatile traffic conditions, the assumption of a particular daily load distribution is crucial in quantifying the relationship between load offered to a trunk group and the average blocking of the trunk group.

APPENDIX A

In Section III, we considered the general class of blocking functions $B(a)$ having the following four properties:

($B1$) $B(0) = 0$ and $B(a) > 0$ for $a > 0$;
($B2$) B is continuous;
($B3$) B is nondecreasing;
($B4$) B is bounded.

We now prove the following theorem:

Theorem 1: Let $\{F_k\}$ be a sequence of probability distribution functions on $[0, \infty)$ *and let $B(a)$ be any blocking function. Then*

$$\lim_{k\to\infty} \int_0^\infty B(a)dF_k(a) = 0 \qquad \text{if and only if}$$

$$\lim_{k\to\infty} F_k(a) = 1 \qquad \text{for} \qquad \text{all } a > 0.$$

Proof: To show sufficiency, let $\epsilon > 0$. Since $B(0) = 0$ and B is continuous, $a < \delta$ implies $B(a) < \epsilon/2$ for sufficiently small δ. Assume that B is bounded by M. Since

$$\lim_{k\to\infty} F_k(a) = 1,$$

for sufficiently large k, $F_k(\delta) \geq 1 - \epsilon/2M$. For such k,

$$\int_0^\infty B(a)dF_k(a) = \int_0^\delta B(a)dF_k(a) + \int_\delta^\infty B(a)dF_k(a) \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus,

$$\lim_{k\to\infty} \int_0^\infty B(a)dF_k(a) = 0.$$

To show necessity, suppose that for some $a_0 > 0$,

$$\lim_{k\to\infty} F_k(a_0) \neq 1.$$

Then there exists a subsequence $\{n_k\}$ and a $\lambda_1 > 0$ such that $F_{n_k}(a_0) \leq 1 - \lambda_1$. Also, ($B1$) to ($B3$) imply that $B(a)$ is bounded away from zero on $[a_0, \infty)$, say by $\lambda_2 > 0$. Then,

$$\int_0^\infty B(a)dF_{n_k}(a) \geq \int_{a_0}^\infty B(a)dF_{n_k}(a) \geq \lambda_2 \int_{a_0}^\infty dF_{n_k}(a) \geq \lambda_2 \cdot \lambda_1 > 0,$$

which contradicts

$$\int_0^\infty B(a)dF_k(a) \to 0;$$

thus,

$$\lim_{k\to\infty} F_k(a) = 1 \quad \text{for all} \quad a > 0.$$

Q.E.D

APPENDIX B

Theorem 2: Let $\{\Gamma(a|\bar{a}, v)\}$ be a family of gamma distributions with given mean $\bar{a}$ and variance v and suppose that $\bar{a} \geq 1$ and $\sqrt{v}/\bar{a} \to \infty$. Then,

$$\Gamma(a|\bar{a}, v) \to 1 \quad \textit{for any} \quad a > 0.$$

Proof: By definition, the gamma distribution with fixed mean $\bar{a}$ and variance v is given by

$$\Gamma(a|\bar{a}, v) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^a t^{\alpha-1} e^{-\beta t} dt, \tag{14}$$

where $\Gamma(\alpha)$ is the gamma function and the parameters α and β are determined by

$$\bar{a} = \alpha/\beta, v = \alpha/\beta^2. \tag{15}$$

Integrating (14) by parts we obtain

$$\Gamma(a|\bar{a}, v) = \frac{\beta^\alpha}{\Gamma(\alpha)\alpha} e^{-\beta t} t^\alpha |_0^a + \frac{\beta^\alpha \beta}{\Gamma(\alpha)\alpha} \int_0^a e^{-\beta t} t^\alpha dt. \tag{16}$$

Let us start by showing that the first term in (16) tends to 1 as the coefficient of variation tends to infinity. From (15) $\alpha = \bar{a}^2/v$ and using the property of the gamma function we obtain

$$\Gamma(\alpha)\alpha = \Gamma(\alpha + 1) \to 1 \quad \text{as} \quad \alpha = \bar{a}^2/v \to 0. \tag{17}$$

From (15) we have

$$\beta^\alpha = \left|\frac{\bar{a}}{v}\right|^{\bar{a}^2/v} = \exp\left|\frac{\bar{a}^2}{v} \ln \frac{\bar{a}}{v}\right|.$$

Using the monotonicity of the exponential and logarithmic functions we have

$$\exp\left|\frac{\bar{a}}{v}\ln\frac{\bar{a}}{v}\right| \le \exp\left|\frac{\bar{a}^2}{v}\ln\frac{\bar{a}}{v}\right| \le \exp\left|\frac{\bar{a}^2}{v}\ln\frac{\bar{a}^2}{v}\right|. \tag{18}$$

Since $\bar{a} \ge 1$ the ratio $\bar{a}/v \to 0$ as $\bar{a}^2/v \to 0$. Thus, passing to the limit in (18) and noting that $\lim_{x\to+0} x\ln(x) = 0$, we obtain that

$$\beta^\alpha \to 1 \quad \text{as} \quad \bar{a}^2/v \to 0. \tag{19}$$

Thus, from (17) and (19) the first term in (14) tends to 1 as $\alpha \to 0$ and $\beta \to 0$.

For the second term in (14) we get

$$\lim_{\substack{\alpha\to 0\\ \beta\to 0}} \frac{\beta^\alpha \beta}{\Gamma(\alpha)\alpha}\int_0^a e^{-\beta t} t^\alpha dt \le \lim_{\substack{\alpha\to 0\\ \beta\to 0}} \frac{\beta^\alpha}{\Gamma(\alpha)\alpha} \lim_{\substack{\alpha\to 0\\ \beta\to 0}} \beta \int_0^a a^\alpha dt = 0. \tag{20}$$

Thus, if $\bar{a} \ge 1$, for any $a > 0$,

$$\Gamma(a\,|\,\bar{a}, v) \to 1 \quad \text{as} \quad \sqrt{v}/a \to \infty.$$

The proof is complete.

We would like to remark that under the assumption $\bar{a} < 1$, Theorem 2 can be reformulated as follows:

$$\Gamma(a\,|\,\bar{a}, v) \to 1 \quad \text{as} \quad v/\bar{a} \to \infty$$

for arbitrary $a > 0$.

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