THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 61, No. 2, February 1982
*Printed in U.S.A.*

# Volterra Expansions for Time-Varying Nonlinear Systems

By I. W. SANDBERG

(Manuscript received August 13, 1981)

*Recent results show for the first time the existence of a locally convergent Volterra-series representation for the input-output relation of a certain important large class of* time-invariant *nonlinear systems containing an arbitrary finite number of nonlinear elements. (Systems of the type considered arise, for example, in the area of communication channel modeling.) Here corresponding results are given for time-varying systems, which arise frequently. A key hypothesis of our main theorem, which asserts that a convergent Volterra expansion exists under certain specified conditions, has the useful property that it is met if a certain "linearized subgraph" of the system is bounded-input bounded-output stable.*

## I. INTRODUCTION

This paper is a continuation of the study initiated in Ref. 1 concerning operator-type models of dynamic nonlinear physical systems, such as communication channels and control systems. Reference 1 addresses the problem of determining conditions under which there exists a power-series-like expansion, or a polynomial-type approximation, for a system's outputs in terms of its inputs. Related problems concerning properties of the expansions are also considered, and nonlocal as well as local results are presented. In particular, the results in Ref. 1 show for the first time the existence of a locally convergent Volterra-series representation for the input-output relation of a certain important large class of *time-invariant* systems containing an arbitrary finite number of nonlinear elements.

The main purpose of this paper is to give corresponding results applicable to time-varying systems, which arise frequently. Also, the results obtained here by specializing to the time-invariant case involve weaker hypotheses concerning the nonlinear elements (here mutual coupling is not ruled out, and, at the expense of somewhat more complicated proofs, we show how to proceed without the local invertibility of a certain mapping associated with the nonlinear elements*).

With regard to background material, functional power series of the form

$$k_0 + \sum_{m=1}^{\infty} \int_a^b \cdots \int_a^b k_m(t, \tau_1, \cdots, \tau_m) u(\tau_1) \cdots u(\tau_m) d\tau_1 \cdots d\tau_m, \quad (1)$$

in which $k_0$ is a constant, $t$ is a parameter, and $u$ and the $k_m$ for $m \geq 1$ are continuous functions, were considered in 1887 by Vito Volterra[2, 3] in connection with his studies of functions of functions (which provided much of the initial motivation to develop the field now known as functional analysis). About twenty years later, Fréchet[4] proved that a continuous real functional (i.e., a continuous real scalar-valued map) defined on a compact set of real continuous functions on $[a, b]$ could be approximated by a sum of a finite number of terms in Volterra's series (1), but with (in analogy with the well-known Weierstrass approximation theorem) the number of terms as well as the $k_m$ dependent on the degree of approximation. Further background material (concerning, in particular, bilinear and polynomic systems) omitted here to avoid unnecessary repetition, can be found in Ref. 1.

Our results are given in the next section, which begins with some mathematical preliminaries followed by a description of the general class of systems to be addressed. Of interest with regard to our main result, Theorem 2 below, is that a key hypothesis has the useful property that it is met if a certain "linearized subgraph" of the system is bounded-input bounded-output stable.

## II. SYSTEMS AND EXPANSIONS

### *2.1 Preliminaries*

Throughout Section II we use $L_\infty(\mathrm{C})$ to denote the complex Banach space of Lebesque measurable complex column $n$-vector-valued functions $v$ defined on the interval $[0, \infty)$ such that the $j$th component $v_j$ of $v$ satisfies $\sup_{t \geq 0} |v_j(t)| < \infty$ for $j = 1, 2, \cdots, n$, and where the norm $\|\cdot\|$ on $L_\infty(\mathrm{C})$ is given by $\|v\| = \max_j \sup_t |v_j(t)|$. (As usual, $n$ denotes an arbitrary positive integer.) The symbol $\theta$ stands for the zero element of $L_\infty(\mathrm{C})$. We use $H(\mathrm{C})$ to denote the linear space of complex column $n$-vector-valued functions $h$ defined on $[0, \infty)$ such that truncations of $h$ belong to $L_\infty(\mathrm{C})$ (i.e., such that $h_{(\omega)} \in L_\infty(\mathrm{C})$ for $\omega \in (0, \infty)$, where

* A simple way to circumvent the need for invertibility mentioned in (Ref. 1, Comments of Section 3.5) is often much less satisfactory for the purpose of obtaining explicit expressions for the Volterra kernels.

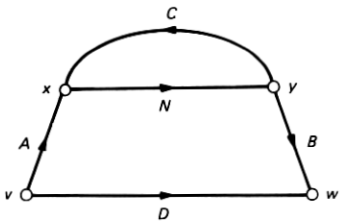


Fig. 1—Signal flow graph.

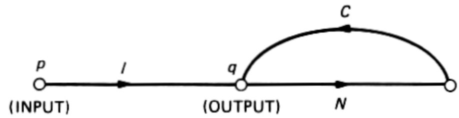


Fig. 2—Feedback part of the flow graph of Fig. 1.

$h_{(\omega)}(t) = h(t)$ for $t \leq \omega$ and $h_{(\omega)}(t) = 0$ otherwise). Clearly unlike $L_\infty(\mathrm{C})$, $\mathrm{H}(\mathrm{C})$ can contain unbounded functions.

### 2.2 The class of systems

Consider a physical system with an input $v$ drawn from $L_\infty(\mathrm{C})$ and an output $w$ contained in $H(\mathrm{C})$. Let the system be composed of linear elements as well as nonlinear elements. Suppose that the nonlinear elements can be viewed as collectively introducing a constraint that can be written as $y = Nx$, in which $N$ is a map from one subset of $H(\mathrm{C})$ into another, where $x$ and $y$, respectively, are the $H(\mathrm{C})$ input and output of the nonlinear part of the system.

With regard to the remainder of the system, which is linear, assume that there are linear maps $A$, $B$, $C$, and $D$ of $H(\mathrm{C})$ into itself such that

$$x = Av + Cy \tag{2}$$

$$w = Dv + By. \tag{3}$$

A signal-flow-graph representation of the relations under consideration is given in Fig. 1.* Concerning the degree of generality of the model, and the assumption that the values of $v$, $w$, $x$, and $y$ have the *same dimension n*, notice that we have not ruled out the possibility that some components of $v$, $x$, and/or $y$ have no effect on the system, and, similarly, that certain of the components of $w$ can be ignored. Nonzero initial conditions, if any, are assumed to be able to be taken into account either in $N$ or as inputs to the system.

### 2.3 General expansions

Consider three hypotheses:

*A.1:* The restrictions of $A$, $B$, $C$, and $D$ to $L_\infty(\mathrm{C})$ are bounded linear maps of $L_\infty(\mathrm{C})$ into itself.

*A.2:* There is an open neighborhood $S_0$ of $\theta$ in $L_\infty(\mathrm{C})$ such that $N$ maps

* This is the same class of systems introduced in Ref. 1. Such representations of systems have been used in different but related settings in Refs. 5, 6, and 7. The maps $A$, $B$, $C$, and $D$ exist for a very large class of systems.

$S_0$ into $L_\infty(C)$ with $N(\theta) = \theta$, and $d^mN$ (the $m$th order Fréchet derivative of $N$) exists on $S_0$ for every $m = 1, 2, \cdots$.

*A.3:* $[I - C_\infty dN(\theta)]$ is an invertible map of $L_\infty(C)$ onto $L_\infty(C)$, in which $I$ is the identity transformation on $L_\infty(C)$, and $C_\infty$ is the restriction of $C$ to $L_\infty(C)$.*

We shall prove the following general result.

*Theorem 1: When A.1, A.2, and A.3 are met, there is a positive number $\delta$ and an open subset $S$ of $S_0$ with the following properties:*

*(i) $\theta \in S$, and for each $v \in L_\infty(C)$ with $\|v\| < \delta$ there exist unique $x$, $y$, and $w$ of $S$, $L_\infty(C)$, and $L_\infty(C)$, respectively, such that* (2), (3), *and $y = Nx$ hold.*

*(ii) The function $w$ described in (i) is given by*

$$w = Dv + \sum_{m=1}^{\infty} B[g_m(Av)]_2 \tag{4}$$

*for $\|v\| < \delta$, in which the $[g_m(Av)]_2$ are defined recursively by the relations*

$$[g_1(Av)]_1 = [I - C_\infty dN(\theta)]^{-1}Av \tag{5}$$

$$[g_1(Av)]_2 = dN(\theta)[g_1(Av)]_1 \tag{6}$$

*and*

$$h_m = \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} d^lN(\theta)[g_{k_1}(Av)]_1 \cdot [g_{k_2}(Av)]_1 \cdots [g_{k_l}(Av)]_1{}^\dagger \tag{7}$$

$$[g_m(Av)]_1 = [I - C_\infty dN(\theta)]^{-1}C_\infty h_m \tag{8}$$

$$[g_m(Av)]_2 = dN(\theta)[g_m(Av)]_1 + h_m \tag{9}$$

*for $m \geq 2$. In addition, the series on the right side of* (4) *converges uniformly with respect to $\|v\| < \delta$.*

#### 2.3.1 Proof of Theorem 1

Notice that (2) and $y = Nx$ can be written as $x - CNx = Av$ and $Nx - y = \theta$, for $y$ and $Nx$ belonging $L_\infty(C)$, and that an expansion for $w$ in terms of $v$ can be obtained at once from (3) and an expansion for $y$ in terms of $v$. These observations motivate us to proceed as follows.‡

---

* Of course, $dN(\theta)$ denotes the Fréchet derivative of $N$ at the point $\theta$.

† In (7), $\sum_{\substack{k_1+k_2+\cdots+k_l=m \\ kj>0}}$ denotes a sum over all positive integers $k_1, \cdots, k_l$ that add to $m$.

‡ An alternative way not pursued here to prove a result along the lines of Theorem 1 involves obtaining an expansion for $x$ in terms of $v$, one for $y$ in terms of $x$, and substituting the former into the latter.

Let $\mathscr{B}$ denote the Banach space $L_\infty(C) \times L_\infty(C)$, whose elements we take to be two-component *column* vectors, normed by $\|u\| = \max(\|u_1\|, \|u_2\|)$ for $(u_1, u_2)' \in L_\infty(C) \times L_\infty(C)$, in which "$'$" denotes transpose. [We use the same symbol for the norms associated with $\mathscr{B}$ and $L_\infty(C)$. The meaning of the symbol will be clear from the context in which it is used.] Let $S_1$ be any open ball in $L_\infty(C)$ of positive radius centered at $\theta$.

Define $F: S_0 \times S_1 \to \mathscr{B}$ to be the map given by

$$F_1(p_1, p_2) = p_1 - C_\infty N p_1$$

$$F_2(p_1, p_2) = N p_1 - p_2$$

for $p \in S_0 \times S_1$.

The set $S_0 \times S_1$ is open in $\mathscr{B}$. By A.2 it easily follows that the derivative $dF(p): \mathscr{B} \to \mathscr{B}$ exists and is continuous at each point $p$ of $S_0 \times S_1$, and that it is given by

$$dF(p) = \begin{pmatrix} [I - C_\infty dN(p_1)] & 0 \\ dN(p_1) & -I \end{pmatrix}, \tag{10}$$

in which here "0" denotes the transformation of $L_\infty(C)$ into itself that replaces each element by $\theta$. By A.3, it follows that $dF(p)$ is an invertible map of $\mathscr{B}$ onto $\mathscr{B}$, with inverse given by

$$dF(p)^{-1} = \begin{pmatrix} [I - C_\infty dN(p_1)]^{-1} & 0 \\ dN(p_1)[I - C_\infty dN(p_1)]^{-1} & -I \end{pmatrix} \tag{11}$$

for $p \in S_0 \times S_1$. Since $dF(p)$ is invertible at $p = (\theta, \theta)'$, by a standard inverse function theorem (Ref. 8, page 273; see also the comment in Ref. 1 concerning Lemma 1 of Ref. 1), there are open neighborhoods $S_2$ and $S_3$ of $(\theta, \theta)'$ in $\mathscr{B}$, with $S_2 \subset S_0 \times S_1$, such that for each $q \in S_3$ there is in $S_2$ a unique $p$ such that $F(p) = q$. Using the boundedness of the restriction of $A$ to $L_\infty(C)$, $\delta_1 > 0$ can be chosen so that $(Av, \theta)' \in S_3$ *for* $v \in L_\infty(C)$ with $\|v\| < \delta_1$, and thus so that for each such $v$, there is in $S_2$ a unique $(x, y)'$ with the property that $F(x, y) = (Av, \theta)'$ [i.e., such that (2) and $y = Nx$ are met].

Observe that the set $S = \{u:(u, Nu)' \in S_2\}$ is an open subset of $S_0$, and that for any $\delta \in (0, \delta_1)$ and for each $v \in L_\infty(C)$ with $\|v\| < \delta$, there is a unique $(x, y, w)$ in $S \times L_\infty(C) \times L_\infty(C)$ such that (2), (3), and $y = Nx$ hold, as claimed in (*i*).

With regard to part (*ii*), we shall use the following Lemma in which $f$ denotes any map from an open subset $X$ of the Banach space $\mathscr{B}$ into $\mathscr{B}$ with the property that there is a nonempty open convex subset $U$ of $\mathscr{B}$ such that for each $u \in U$ there is in $X$ a unique $x_u$ such that $f(x_u) = u$, and in which $g$ stands for the map of $U$ into $X$ defined by $f[g(u)] = u$ for $u \in U$.

*Lemma* 1: *Assume that the Fréchet derivative* $d^m f$ *exists on* $X$ *for each* $m$. *Let* $u_0 \in U$, *and suppose that* $df[g(u_0)]$ *is an invertible map of* $\mathscr{B}$ *onto itself. Then there is a* $\sigma > 0$ *such that the expansion*

$$g(u) = g(u_0) + \sum_{m=1}^{\infty} g_m(u_0, u - u_0)$$

*is valid and uniformly convergent for* $u \in U$ *with* $\|u - u_0\| < \sigma$, *where*

$$g_1(u_0, u - u_0) = df[g(u_0)]^{-1}(u - u_0),$$

*and*

$$g_m(u_0, u - u_0) = -df[g(u_0)]^{-1} \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} d^l f[g(u_0)]$$

$$\cdot g_{k_1}(u_0, u - u_0) g_{k_2}(u_0, u - u_0) \cdots g_{k_l}(u_0, u - u_0), \qquad m \geq 2.$$

Lemma 1 is a special case of Theorem 4 of Ref. 1 (see also Ref. 9).

With $S_2$ and $S_3$ as indicated above before Lemma 1, choose $X = S_2$, assume without loss of generality that $S_3$ is convex, and take $U = S_3$. Throughout the remainder of this section, let $f$ denote the restriction of $F$ to $X$. The following lemma is proved in Appendix A.

*Lemma* 2: *Under the conditions of Theorem* 1, *for each* $p \in X$ *and every* $l = 2, 3, \cdots$ *the* $l$th *order Fréchet derivative* $d^l f(p)$ *exists, and we have*

$$d^l f(p) h_1 h_2 \cdots h_l = \begin{pmatrix} -C_\infty d^l N(p_1) h_{11} h_{21} \cdots h_{l1} \\ d^l N(p_1) h_{11} h_{21} \cdots h_{l1} \end{pmatrix} \tag{12}$$

*for any elements* $h_1, h_2, \cdots, h_l$ *of* $\mathscr{B}$ (*where* $h_{j1}$ *denotes the first component of* $h_j$ *for each* $j$).

By Lemmas 1 and 2, there is a $\sigma > 0$ such that $S_3$ contains an open ball in $\mathscr{B}$ centered at $(\theta, \theta)'$ of radius $\sigma$, and the solution $r \in X$ of $f(r) = s$ for $s \in \mathscr{B}$ with $\|s\| < \sigma$ is given by the uniformly convergent series $\sum_{m=1}^{\infty} g_m(s)$, in which $g_1(s) = df[(\theta, \theta)']^{-1}s$, and

$$g_m(s) = -dF[(\theta, \theta)']^{-1} \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}}$$

$$\cdot d^{\mathscr{L}} f[(\theta, \theta)'] g_{k_1}(s) \cdots g_{k_l}(s)$$

for $m \geq 2$. In particular, by (11) and Lemma 2, when $s \in B$ with $\|s\| < \sigma$ and $s$ has the form $(s_1, \theta)'$, we have

$$g_1(s) = \begin{pmatrix} [I - C_\infty dN(\theta)]^{-1} s_1 \\ dN(\theta)[I - C_\infty dN(\theta)]^{-1} s_1 \end{pmatrix} \tag{13}$$

and

$$g_m(s) = -\begin{pmatrix} [I - C_\infty dN(\theta)]^{-1} & 0 \\ dN(\theta)[I - C_\infty dN(\theta)]^{-1} & -I \end{pmatrix} \sum_{l=2}^{m} (l!)^{-1}$$

$$\cdot \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} \begin{pmatrix} -C_\infty d^l N(\theta)[g_{k_1}(s)]_1 \cdots [g_{k_l}(s)]_1 \\ d^l N(\theta)[g_{k_1}(s)]_1 \cdots [g_{k_l}(s)]_1 \end{pmatrix} \quad (14)$$

for $m \geq 2$.

Now choose $\delta \in (0, \delta_1)$ so that $\|v\| < \delta$ implies that $\|Av\| < \sigma$, and, referring to the $g_m$ of (13) and (14), observe that for $\|v\| < \delta$, $y$ of part $(i)$ of the theorem is given by $y = \sum_{m=1}^{\infty} \{g_m[(Av, \theta)']\}_2$. From (14),

$$[g_m(s)]_1 = [I - C_\infty dN(\theta)]^{-1} \cdot \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} C_\infty d^l N(\theta)[g_{k_1}(s)]_1 \cdots [g_{k_l}(s)]_1,$$

and

$$[g_m(s)]_2 = dN(\theta)[I - C_\infty dN(\theta)]^{-1} \cdot \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} C_\infty d^l N(\theta)[g_{k_1}(s)]_1 \cdots [g_{k_l}(s)]_1 + \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} d^l N(\theta)[g_{k_1}(s)]_1 \cdots [g_{k_l}(s)]_1$$

for $m \geq 2$ and $\|s_1\| < \sigma$ which, together with (13), completes the proof of the theorem. (The $[g_m(Av)]_i$ in Theorem 1 correspond to the $\{g_m[(Av, \theta)']\}_i$ here.)

### 2.3.2 Comments

In principle, it is straightforward to give an explicit expression for any term in the series in (4). For example, it is a simple exercise to verify that the third-order term $B[g_3(Av)]_2$ is

$$\tfrac{1}{6} B\{dN(\theta)[I - C_\infty dN(\theta)]^{-1} C_\infty + I\}\, d^3 N(\theta) \cdot \{[I - C_\infty dN(\theta)]^{-1} Av\}^3, \quad (15)$$

when $d^2N(\theta)$ is the zero operator (i.e., is the zero operator in the space to which $d^2N(\theta)$ belongs). If $dN(\theta)$ also is the zero operator, then of course (15) is simply

$$\tfrac{1}{6} B d^3 N(\theta)(Av)^3.$$

The interpretation of (15), and more general expressions, under certain assumptions concerning the forms of $N$, $A$, $B$, and $C$ is addressed in the following section.

Theorem 1 (and Lemma 1) hold if $L_\infty(\mathrm{C})$ and $H(\mathrm{C})$, respectively, are replaced with any complex Banach space and any linear space containing the elements of the complex Banach space.

### 2.4 Volterra expansions

In this section, we introduce, discuss, and prove our main result. We begin by considering the following definitions and two hypotheses, B.1 and B.2.

For each $l = 1, 2, \cdots$, let $R_0(l)$ denote the subset of $\mathrm{R}^{(l+1)}$ given by $R_0(l) = \{(v_0, v_1, \cdots, v_l) \in \mathrm{R}^{(l+1)}: v_0 \geq v_i \geq 0 \text{ for } i = 1, 2, \cdots, l\}$.

For any positive integers $q$ and $l$, let $S_q^{(l)}$ denote the set of complex $n \times q$ matrix-valued functions $h$ defined on $R_0(l)$ such that each $h_{ij}$ is Lebesgue measurable and bounded on $R_0(l)$, and satisfies

$$\sup_{t \geq 0} \int_{[0,t]^l} |h_{ij}(t, \tau_1, \cdots, \tau_l)| d(\tau_1, \cdots, \tau_l) < \infty. \tag{16}$$

*B.1:* There are elements $a$, $b$, $c$, and $d$ of $S_n^{(1)}$ such that for each $p \in H(\mathrm{C})$,

$$(Ap)(t) = \int_0^t a(t, \tau)p(\tau)d\tau$$

$$(Bp)(t) = \int_0^t b(t, \tau)p(\tau)d\tau$$

$$(Cp)(t) = \int_0^t c(t, \tau)p(\tau)d\tau$$

$$(Dp)(t) = \int_0^t d(t, \tau)p(\tau)d\tau$$

for $t \geq 0$.

In hypothesis B.2 below, $\Gamma_0$ denotes the set $\{z \in \mathrm{C}^n: |z_i| < \gamma \text{ for } i = 1, 2, \cdots, n\}$, in which $\gamma$ is a positive constant and $\mathrm{C}^n$ is the normed linear space of complex column $n$-vectors with zero element $\theta_\mathrm{C}$ and norm $|\cdot|$ given by $|z| = \max_i |z_i|$ for $z \in \mathrm{C}^n$.

*B.2:* $N$ is defined on $\Gamma = \{s \in L_\infty(\mathrm{C}): \|s\| < \gamma\}$ by

$$(Ns)(t) = \eta[s(t), t], \qquad t \geq 0$$

where $\eta$ is a map from $\Gamma_0 \times [0, \infty)$ into $\mathrm{C}^n$ with the following properties:

(*i*) $\eta(\theta_\mathrm{C}, t) = \theta_\mathrm{C}$ for $t \geq 0$.

(*ii*) The function $\xi$ given by $\xi(t) = \eta[s(t), t]$, $t \geq 0$ is Lebesgue measurable on $[0, \infty)$ for each $s \in \Gamma$.

(*iii*) For each $t \in [0, \infty)$, $\eta(\cdot, t)$ is a continuous map of $\Gamma_0$ into $C^n$, and for each $t \in [0, \infty)$, for $1 \le i, j \le n$, and for any point $\alpha \in \Gamma_0$, the function $z_j \mapsto \eta_i(\alpha_1, \cdots, \alpha_{j-1}, z_j, \alpha_{j+1}, \cdots, \alpha_n, t)$ is differentiable with respect to the complex variable $z_j$ for $|z_j| < \gamma$. [This implies (see Ref. 8, pages 204, 205, 226, 227, 230) the existence throughout $\Gamma_0$ of every $m$th order partial derivative

$$\frac{\partial^m \eta_i}{\partial z_{j_m} \partial z_{j_{m-1}} \cdots \partial z_{j_1}} \tag{17}$$

for each $t$ and all $m$.]

(*iv*) For any $m, j_1, \cdots, j_m$, and $i$, the partial derivative (17), which we denote by $p(z_1, \cdots, z_n, t)$, satisfies the conditions that the function $t \mapsto p(0, \cdots, 0, t)$ is bounded on $[0, \infty)$, and that $p$ is uniformly continuous on closed subsets of $\Gamma_0$ uniformly in $t$, in the sense that given a closed $\Gamma_{00} \subset \Gamma_0$ and a $\delta_1 > 0$ there is a $\delta_2 > 0$ such that

$$|p(z_{a1}, \cdots, z_{an}, t) - p(z_{b1}, \cdots, z_{bn}, t)| < \delta_1$$

for $t \ge 0$ whenever $z_a$ and $z_b$ are elements of $\Gamma_{00}$ such that $|z_a - z_b| < \delta_2$.

Following are comments and an example.

If $\eta(\cdot, t)$ is independent of $t$ and (*iii*) is met, then (*ii*) and (*iv*) are met.

The conditions on $\eta$ of B.2 are met if, for example,

$$\eta_i(z, t) = \sum_{j=1}^{\rho} \beta_{ij}(t)\lambda_{ij}(z_i), \qquad t \ge 0$$

for each $i$ and $z \in \Gamma_0$, in which $\rho$ is a positive integer, the $\beta_{ij}$ are $C^1$-valued bounded measurable functions, and each $\lambda_{ij}$ is an analytic function from the disk $|z_i| < \gamma$ in $C^1$ into $C^1$ such that $\lambda_{ij}(0) = 0$. In this important case, $N$ restricted to $\Gamma$ can of course be represented by $n$ single-input single-output memoryless, possibly time-varying, nonlinear operators.

In order to introduce another needed hypothesis, consider the following proposition.

*Proposition 1: When $c \in S_n^{(1)}$ and $\eta$ satisfies the conditions of B.2, for each $p \in H(C)$ there exists a unique $q \in H(C)$ such that*

$$p(t) = q(t) - \int_0^t c(t, \tau)L(\tau)q(\tau)d\tau, \qquad t \ge 0, \tag{18}$$

*where $L$ is the $n \times n$ matrix-valued function defined on $[0, \infty)$ by $L_{ij}(t) = \partial\eta_i(z_1, \cdots, z_n, t)/\partial z_j$ at $z_1 = z_2 = \cdots = z_n = 0$ for each $i, j$, and $t$.*

Since $L$ is measurable on $[0, \infty)$ (see the proof of Lemma 3 in

Appendix B), Proposition 1 follows at once from Lemma 6 in Section 2.4.2.

*B.3:* Under the hypotheses of Proposition 1, (18) has the further property that $p \in L_\infty(C)$ implies that the solution $q$ also belongs to $L_\infty(C)$.

The "further property" of B.3 has the interpretation that the feedback part of the graph of Fig. 1, shown in Fig. 2, is *bounded-input bounded-output stable* in the indicated sense when $N$ is replaced with its linearization at the origin [by which we mean its linearization (see Lemma 3 below) at $\theta$ extended in the natural way to all of $H(C)$]. The node labeled "output" in Fig. 2 is an intermediate output node. For our purposes here, the output label can be moved to the node to the right of $N$ when the matrix $L(t)^{-1}$ exists for each $t \geq 0$ and has uniformly bounded elements.

We shall use also the following definition and proposition:

*Definition*: Throughout the remainder of this section, for each $l$, $\chi[v(\tau_1), \cdots, v(\tau_l)]$ denotes the column vector of order $n^l$ whose elements are the $n^l$ distinct products $v_{\omega_1}(\tau_1)v_{\omega_2}(\tau_2) \cdots v_{\omega_l}(\tau_l)$, corresponding to distinct sequences $\omega_1, \omega_2, \cdots, \omega_l$ with each $\omega_j$ drawn from $\{1, 2, \cdots, n\}$, arranged in an arbitrary predetermined order.

*Proposition 2: If $k_l \in S_{n^l}^{(l)}$ for some $l$, then the iterated integral*

$$\int_0^t \cdots \int_0^t k_l(t, \tau_1, \cdots, \tau_l)\chi[v(\tau_1), \cdots, v(\tau_l)]d\tau_1 \cdots d\tau_l$$

*exists and is invariant with respect to interchanges in the order of integration for each $t \geq 0$ and $v \in L_\infty(C)$, and $V_{k_l}(v)$, defined on $[0, \infty)$ by*

$$V_{k_l}(v)(t) = \int_0^t \cdots \int_0^t k_l(t, \tau_1, \cdots, \tau_l)\chi[v(\tau_1), \cdots, v(\tau_l)]d\tau_1 \cdots d\tau_l$$

*for an arbitrary $v \in L_\infty(C)$, is an element of $L_\infty(C)$.*

Proposition 2 is a special case of Lemma 4 of Section 2.4.2.

The following is our main result.

*Theorem 2: Suppose that B.1, B.2, and B.3 are met. Then*

*(i) The hypotheses of Theorem 1 are satisfied.*

*(ii) For each $l = 1, 2, \cdots$ there is a $k_l \in S_{n^l}^{(l)}$ such that*

$$w = \sum_{l=1}^{\infty} V_{k_l}(v) \quad \text{for} \quad \|v\| < \delta, \tag{19}$$

*with the series uniformly convergent with respect to $\|v\| < \delta$, where $v$, $w$, and $\delta$ are described in Theorem 1, and $V_{k_l}(\cdot)$ is as indicated in Proposition 2.*

*(iii) Each $k_l$ can be taken to be continuous on $R_0(l)$ when $a$ and $d$*

*are continuous on $R_0(1)$, and b and c meet the condition imposed on s in part (ii) of Lemma 4 below [the condition is met if b and c are continuous on $R_0(1)$].*

### 2.4.1 Comments

Theorem 2 is proved in the next section. Using the proof given there, it can be shown that, as one would expect, the Volterra kernels $k_l$ can be taken to depend on only $(t - \tau_1), \cdots, (t - \tau_l)$ when $a$, $b$, $c$, and $d$ depend only on $(t - \tau)$, and $\eta(\cdot, t)$ is independent of $t$.*

Similarly, each $k_l$ can be taken to be real valued (i.e., to have zero imaginary part) if $a$, $b$, $c$, $d$, and the partial derivatives of $\eta(\cdot, t)$ at the origin are real valued. This shows that Theorem 2 establishes the existence of a Volterra-series expansion for the important corresponding case in which $v$, $w$, $x$, and $y$ in Fig. 1 are restricted to be real valued and $N$ (which then would be a map between real-valued function spaces) can be analytically extended so that the hypotheses of the theorem are met.†

For the single-input case in which either $n = 1$ or $v_i(t) = 0$ for all $t$ and $i = 2, 3, \cdots, n$, (19) takes the more familiar form

$$w(t) = \sum_{l=1}^{\infty} \int_0^t \cdots \int_0^t h_l(t, \tau_1, \cdots, \tau_l) \cdot v_1(\tau_1)v_1(\tau_2) \cdots v_1(\tau_l)d\tau_1 d\tau_2 \cdots d\tau_l, \qquad t \geq 0$$

for $\sup_{t \geq 0}|v_1(t)| < \delta$, with the $h_l$ belonging to $S_1^{(l)}$.

By modifying the proof given in Section 2.4.2, results similar to Theorem 2 can be obtained for cases in which the basic underlying function space $L_\infty(\mathrm{C})$ is replaced with another complex Banach space, and/or $A$, $B$, $C$, and $D$ have a more general‡ (or different) form. Of some importance is the case in which $L_\infty(\mathrm{C})$ is replaced with the corresponding set $L_\infty(\mathrm{C})(T)$ of bounded functions defined on a finite interval $[0, T]$, and a theorem along the lines of Theorem 2 for this case is given in Appendix F.

### 2.4.2 Proof of Theorem 2

Our proof uses five lemmas, which are proved in the appendix, and an inductive argument using Theorem 1. We begin with a description of the lemmas and some associated definitions.

* See Proposition 7 and Lemma 2 of Ref. 1.

† In this connection, Theorem 5 of Ref. 1 can be used in place of Lemma 1 to prove results along the same lines as Theorems 1 and 2, but with $L_\infty(\mathrm{C})$ replaced with the corresponding function space over the *real* field, and Corollary 1 of Ref. 1 can be used to obtain corresponding $p$th order *approximation* results under weaker differentiability hypotheses.

‡ Detailed results for cases in which $a$, $b$, $c$, and $d$ are replaced with certain generalized functions, and $N$ is not necessarily memoryless, will be given in another paper.

*Lemma* 3: *Suppose that B.2 is met. Then N maps* $\Gamma$ *into* $L_\infty(C)$, *each*

$$\frac{\partial^m \eta_i[s_1(\cdot), \cdots, s_n(\cdot), \cdot]}{\partial z_{j_m} \partial z_{j_{m-1}} \cdots \partial z_{j_1}}$$

*is bounded and measurable on* $[0, \infty)$ *for each* $s \in \Gamma$, $d^m N(s)$ *exists for each* $s \in \Gamma$ *and all* $m = 1, 2, \cdots$, *and, for any m, we have* $[d^m N(s) h_1 \cdots h_m(t)]_i =$

$$\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} \frac{\partial^m \eta_i[s_1(t), \cdots, s_n(t), t]}{\partial z_{j_m} \partial z_{j_{m-1}} \cdots \partial z_{j_1}} h_{1j_1}(t) h_{2j_2}(t) \cdots h_{mj_m}(t), \; t \geq 0$$

*for each* $s \in \Gamma$, *each i, and any m elements* $h_1, h_2, \cdots, h_m$ *of* $L_\infty(C)$.

Definition: For each $h \in S_q^{(l)}$, $\tilde{h}$ denotes the function defined on $[0, \infty)^{(l+1)}$ by $\tilde{h} = h$ on $R_0(l)$ and $h = 0_{nq}$ (the zero $n \times q$ matrix) otherwise.

*Lemma* 4: *Suppose that* $h \in S_1^{(l)}$ *for some* $l \geq 1$, *that* $s \in S_n^{(1)}$, *and that u is a bounded measurable function from* $[0, \infty)^l$ *into the complex numbers. Then*

(*i*) *The function k defined by*

$$k(t, \tau_1, \cdots, \tau_l) = \int_0^t s(t, \tau) \tilde{h}(\tau, \tau_1, \cdots, \tau_l) d\tau$$

*for* $(t, \tau_1, \cdots, \tau_l) \in R_0(l)$, *belongs to* $S_1^{(l)}$.

(*ii*) *If h is continuous on* $R_0(l)$, *and* $\tilde{s}$ *meets the condition that each* $\delta_{ij}$, *given by*

$$\delta_{ij}(\alpha, t) = \int_0^\infty |\tilde{s}_{ij}(t + \alpha, \tau) - \tilde{s}_{ij}(t, \tau)| d\tau$$

*for* $t \geq 0$ *and* $(t + \alpha) \geq 0$, *satisfies* $\delta_{ij}(\alpha, t) \to 0$ *as* $\alpha \to 0$ *for each t, then k is continuous on* $R_0(l)$.

(*iii*) *The iterated integrals*

$$\int_0^t \cdots \int_0^t h(t, \tau_1, \cdots, \tau_l) u(\tau_1, \cdots, \tau_l) d\tau_1 \cdots d\tau_l$$

*and*

$$\int_0^t \cdots \int_0^t k(t, \tau_1, \cdots \tau_l) u(\tau_1, \cdots, \tau_l) d\tau_1 \cdots d\tau_l$$

*exist, and are invariant with respect to interchanges of orders of integration, for* $t \geq 0$, *and p defined by*

$$p(t) = \int_0^t \cdots \int_0^t h(t, \tau_1, \cdots, \tau_l) u(\tau_1, \cdots, \tau_l) d\tau_1 \cdots d\tau_l, \qquad t \geq 0$$

*is an element of* $L_\infty(C)$.

(*iv*) *We have*

$$\int_0^t s(t, \tau) \int_0^\tau \cdots \int_0^\tau h(\tau, \tau_1, \cdots, \tau_l) u(\tau_1, \cdots, \tau_l) d\tau_1 \cdots d\tau_l d\tau$$
$$= \int_0^t \cdots \int_0^t k(t, \tau_1, \cdots, \tau_l) u(\tau_1, \cdots, \tau_l) d\tau_1 \cdots d\tau_l, \qquad t \geq 0.$$

**Comment**

The condition on $s$ of part (*ii*) of Lemma 4 is met if $s$ is continuous on $R_0(1)$, or if $s(t, \tau)$ depends only on the difference $(t - \tau)$ (see Ref. 11, page 12).

*Definition*: If $r$ and $s$ are two complex column $n$-vectors, then $rs$ denotes the column $n$-vector defined by $(rs)_i = r_i s_i$ for $i = 1, 2, \cdots, n$.

*Lemma* 5: *If* $h \in S_1^{(p)}$ *and* $k \in S_1^{(q)}$, *then the function* $s$, *defined on* $R_0(p + q)$ *by*

$$s(t, \tau_1, \cdots, \tau_{p+q}) = h(t, \tau_1, \cdots, \tau_p) k(t, \tau_{p+1}, \cdots, \tau_{p+q})$$

*for* $(t, \tau_1, \cdots, \tau_{p+q}) \in R_0(p + q)$, *belongs to* $S_1^{(p+q)}$.

*Lemma* 6: *If* $\lambda \in S_n^{(1)}$, *then for each* $p \in H(C)$ *there is a unique* $q \in H(C)$ *such that*

$$p(t) = q(t) - \int_0^t \lambda(t, \tau) q(\tau) d\tau, \qquad t \geq 0. \tag{20}$$

In Lemma 7, below, we refer to the following two hypotheses.

*C.1:* $\lambda \in S_n^{(1)}$, and $\Lambda$ denotes the map of $L_\infty(C)$ into itself defined by

$$(\Lambda p)(t) = \int_0^t \lambda(t, \tau) p(\tau) d\tau, \qquad t \geq 0$$

for $p \in L_\infty(C)$.

*C.2:* $\lambda \in S_n^{(1)}$, and, for each $p \in L_\infty(C)$, the unique element $q$ of $H(C)$ such that

$$p(t) = q(t) - \int_0^t \lambda(t, \tau) q(\tau) d\tau, \qquad t \geq 0$$

satisfies the condition that $q \in L_\infty(C)$.

*Lemma* 7: *Suppose that* C.1 *and* C.2 *hold. Then* $(I - \Lambda)$ *is an invertible map* of* $L_\infty(C)$ *onto itself, and there is a* $\kappa \in S_n^{(1)}$ *such that*

* Here, as in Section 2.3, $I$ denotes the identity transformation on $L_\infty(C)$.

$$(I-\Lambda)^{-1}p(t) = p(t) - \int_0^t \kappa(t, \tau)p(\tau)d\tau, \qquad t \geq 0$$

*for every $p \in L_\infty(C)$, and such that if $\lambda$ meets the conditions imposed on s of part (ii) of Lemma 4, then so does $\kappa$.*

This concludes our statement of the lemmas that we shall use. As mentioned at the beginning of this section, Lemmas 3 through 7 are proved in the appendix.

It is clear that (under the hypotheses of Theorem 2) A.1 is met, Lemma 3 shows that A.2 is satisfied, and, by Lemmas 3 and 6, as well as the observation that C.1 together with C.2 imply that $(I - \Lambda)^{-1}$ exists, we see that A.3 also is satisfied. Therefore, the hypotheses of Theorem 1 are met.

With $v$, $w$, and $\delta$ as in part (*ii*) of Theorem 1,

$$w = Dv + \sum_{m=1}^{\infty} B[g_m(Av)]_2$$

for $\|v\| < \delta$, where the $[g_m(Av)]_2$ are defined by (5) through (9), which involve associated functions $[g_m(Av)]_1$.

For each positive integer $p$, let $H_p$ denote the hypothesis that we have

$$[g_m(Av)]_1(t) = \int_0^t \cdots \int_0^t q_m(t, \tau_1, \cdots \tau_m) \cdot \chi[v(\tau_1), \cdots, v(\tau_m)]d\tau_1 \cdots d\tau_m,$$

and

$$[g_m(Av)]_2(t) = L(t)[g_m(Av)]_1(t) + \sum_{l=2}^{m} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=m \\ k_j>0}} B_l(t) \int_0^t \cdots \int_0^t r_{k_1,\cdots,k_l}(t, \tau_1, \cdots, \tau_m) \cdot \chi[v(\tau_1), \cdots, v(\tau_m)]d\tau_1 \cdots d\tau_m$$

for $t \geq 0$, $\|v\| < \delta$, and $m = 1, 2, \cdots, p$, in which

(*i*) by the sum over $l$ when $m = 1$ is meant the zero $n$-vector,

(*ii*) each $q_m$ belongs to $S_{n^{(m)}}^{(m)}$,

(*iii*) $L$ is the $n \times n$ matrix-valued function described in Proposition 1,

(*iv*) for $l \geq 2$, the $B_l$ are bounded* measurable $n \times n^l$ matrix-valued functions over $[0, \infty)$,

(*v*) for $m \geq 2$, the $r_{k_1,\cdots,k_l}$ are $n^l \times n^m$ matrix-valued functions

* By $B_l$ bounded is meant that its elements are bounded.

defined on $R_0(m)$ such that each $(r_{k_1,\ldots,k_l})_{ij} \in S_1^{(m)}$ with $n = 1$, and

(*vi*) the $q_m$, and the $r_{k_1,\ldots,k_l}$ for $m \geq 2$, are continuous on $R_0(m)$ when $a$ is continuous on $R_0(1)$ and $c$ meets the conditions on $s$ of part (*ii*) of Lemma 4.

By Lemma 3, $L$ is bounded and measurable. Using (5) and (6) as well as Lemmas 3, 4, and 7, we see that $H_1$ is met. [Notice that $s$ given by $s(t, \tau) = u(t, \tau)v(\tau)$ meets the condition of part (*ii*) of Lemma 4 when $u \in S_n^{(1)}$, $u$ meets the condition, and $v$ is a bounded measurable $n \times n$ matrix-valued function on $[0, \infty)$.] Thus, by Lemma 4, there is a $k_1 \in S_n^{(1)}$ such that

$$Dv(t) + B[g_1(Av)]_2(t) = \int_0^t k_1(t, \tau)v(\tau)d\tau, \qquad t \geq 0$$

for $\|v\| < \delta$, and $k_1$ is continuous under the conditions on $a$, $b$, $c$, and $d$ of part (*iii*) of Theorem 2.

By Lemma 4 (which holds for any $n$), it easily follows that if $H_p$ is met for some $p \geq 2$ then there is a $k_p \in S_{n^p}^{(p)}$ such that

$$B[g_p(Av)]_2(t) = \int_0^t \cdots \int_0^t k_p(t, \tau_1, \cdots, \tau_p) \cdot \chi[v(\tau_1), \cdots, v(\tau_p)]d\tau_1 \cdots d\tau_p, \qquad t \geq 0$$

for $\|v\| < \delta$, and such that $k_p$ meets the continuity requirement of part (*iii*) of the theorem. Therefore, to complete the proof of the theorem it suffices to show that $H_p$ is met for every $p$. For this purpose, suppose that $H_p$ is satisfied for some $p$. Using (7), we have

$$h_{(p+1)} = \sum_{l=2}^{(p+1)} (l!)^{-1} \sum_{\substack{k_1+k_2+\cdots+k_l=(p+1)\\ k_j>0}} d^lN(\theta) \cdot [g_{k_1}(Av)]_1[g_{k_2}(Av)]_1 \cdots [g_{k_l}(Av)]_1, \qquad \|v\| < \delta.$$

Now let $l$ be a fixed integer such that $2 \leq l \leq (p + 1)$, and let $k_1, \cdots, k_l$ be positive integers such that $k_1 + k_2 + \cdots + k_l = p + 1$. Using Lemma 3,

$$\{d^lN(\theta)[g_{k_1}(Av)]_1[g_{k_2}(Av)]_1 \cdots [g_{k_l}(Av)]_1(t)\}_i$$
$$= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_l=1}^{n} b_i(t, j_1, \cdots, j_l) \left[\int_0^t \cdots \int_0^t q_{k_1}(t, \tau_1, \cdots, \tau_{k_1})\chi \cdot [v(\tau_1), \cdots, v(\tau_{k_1})]d\tau_1 \cdots d\tau_{k_1}\right]_{j_1} \cdots \left[\int_0^t \cdots \int_0^t q_{k_l}(t, \tau_1, \cdots \tau_{k_l}) \cdot \chi[v(\tau_1), \cdots, v(\tau_{k_l})]d\tau_1 \cdots d\tau_{k_l}\right]_{j_l}$$

for $t \geq 0$, $\|v\| < \delta$, and each $i$, in which the $b_i(\cdot, j_1, \cdots, j_l)$ are bounded and measurable. By Lemmas 4 and 5, we see that

$$d^l N(\theta)[g_{k_1}(Av)]_1 \cdots [g_{k_l}(Av)]_1(t)$$

$$= B_l(t) \int_0^t \cdots \int_0^t r_{k_1,\cdots,k_l}(t, \tau_1, \cdots, \tau_{(p+1)})$$

$$\cdot \chi[v(\tau_1), \cdots, v(\tau_{(p+1)})] d\tau_1 \cdots \tau_{(p+1)}, \qquad t \geq 0$$

for $\|v\| < \delta$ and for some $B_l$ and $r_{k_1,\cdots,k_l}$ of the type required. [Here we have used the observations that a product of integrals

$$\int_0^t \cdots \int_0^t q_{k_1}(t, \tau_1, \cdots, \tau_{k_1})_{j_1 l_1} \{\chi[v(\tau_1), \cdots, v(\tau_{k_1})]\}_{l_1}$$

$$\cdot d\tau_1 \cdots d\tau_{k_1} \cdots \int_0^t \cdots \int_0^t q_{k_l}(t, \tau_1, \cdots, \tau_{k_l})_{j_l l_l}$$

$$\cdot \{\chi[v(\tau_1), \cdots, v(\tau_{k_l})]\}_{l_l} d\tau_1 \cdots d\tau_{k_l},$$

in which $l_j$ is drawn from $\{1, 2, \cdots, n^{k_j}\}$ for each $j$, can be written as the iterated integral

$$\int_0^t \cdots \int_0^t q_{k_1}(t, \tau_1, \cdots, \tau_{k_1})_{j_1 l_1} \cdots q_{k_l}(t, \tau_{(k_1 + \cdots + k_{l-1} + 1)}, \cdots,$$

$$\cdot \tau_{(p+1)})_{j_l l_l} \{\chi[v(\tau_1), \cdots, v(\tau_{k_1})]\}_{l_1} \cdots$$

$$\cdot \{\chi[v(\tau_{(k_1 + \cdots + k_{l-1} + 1)}), \cdots, v(\tau_{(p+1)})]\}_{l_l} d\tau_1 \cdots d\tau_{(p+1)},$$

and that $r$, given by $r(t, \tau_1, \cdots, \tau_{p+1}) = q_{k_1}(t, \tau_1, \cdots, \tau_{k_1}) \cdot {}_{j_1 l_1} \cdots q_{k_l}(t, \tau_{(k_1 + \cdots + k_{l-1} + 1)}, \cdots, \tau_{p+1})_{j_l l_l}$ on $R_0(p+1)$, is continuous when each $q_{k_j}$ is continuous on $R_0(k_j)$.]

Finally, using (8) and (9), and Lemmas 4 and 7, we observe that $H_{(p+1)}$ is satisfied, showing that $H_p$ is met for all $p$. This completes the proof.*

## APPENDIX A

### *Proof of Lemma 2*

Assume that $p \in X$ is given.

Let $Q$ denote the linear map from $\mathscr{B}$ into the space $L(\mathscr{B}, \mathscr{B})$ of bounded linear operators from $\mathscr{B}$ into $\mathscr{B}$, given by

* Our proof shows also that the theorem holds if B.1 and B.2 are modified to the extent that an arbitrary constant (scalar or $n \times n$-matrix) multiple of the identity map in $H$(C) is added to $B$, and $\eta(\cdot, t)$ is required to be independent of $t$.

$$Qr = \begin{bmatrix} -C_\infty d^2N(p_1)r_1 & 0 \\ d^2N(p_1)r_1 & 0 \end{bmatrix} \tag{21}$$

for any $r \in \mathscr{B}$. Since $\sup\{\|Qh_1h_2\|: h_1, h_2 \in \mathscr{B} \text{ with } \|h_1\| = \|h_2\| = 1\}$ is finite, it follows that $Q$ is bounded.

Let $h \in \mathscr{B}$ be such that $(p + h) \in X$. Observe that, using (10) and (21),

$$\|df(p + h) - df(p) - Qh\| = \sup\{\|df(p + h)h_1 - df(p)h_1 - Qhh_1\|: h_1 \in B, \|h_1\| = 1\} = o(\|h\|),$$

which shows that $d^2f(p)$ exists, that $d^2f(p) = Q$, and hence that the expression for $d^2f(p)h_1h_2$ given in the lemma is valid.

Now suppose that for some $l \geq 2$, $d^lf(p)$ exists and that it satisfies (12). Met $\tilde{M}$ denote the continuous multilinear mapping of $\mathscr{B}^{(l+1)}$ into $\mathscr{B}$ given by

$$\tilde{M}(q_1, q_2, \cdots, q_{(l+1)}) = \begin{bmatrix} -C_\infty d^{(l+1)}N(p_1)q_{11}q_{21} \cdots q_{(l+1)1} \\ d^{(l+1)}N(p_1)q_{11}q_{21} \cdots q_{(l+1)1} \end{bmatrix}$$

for $q_1, q_2, \cdots, q_{(l+1)}$ belonging to $\mathscr{B}$. We shall use $M$ to denote the usual associate (Ref. 10, page 318) of $\tilde{M}$ that belongs to $L(\mathscr{B}, L(\mathscr{B}, \cdots, L(\mathscr{B}, \mathscr{B})\cdots))$ with $(l + 1)$ $L$'s, in which $L(A_1, A_2)$ stands for the set of continuous linear operators from the Banach space $A_1$ into the Banach space $A_2$.*

Using the fact that

$$\|d^lf(p + h) - d^lf(p) - Mh\| = \sup\{\|d^lf(p + h) \cdot h_1h_2 \cdots h_l - d^lf(p)h_1h_2 \cdots h_l - Mhh_1h_2 \cdots h_l\|: \|h_1\| = \|h_2\| = \cdots = \|h_l\| = 1\}$$

for $(p + h) \in X$, as well as the boundedness of $C_\infty$, we find that $\|d^lf(p + h) - d^lf(p) - Mh\| = o(\|h\|)$ as $\|h\| \to 0$, which shows that $d^{(l+1)}f(p)$ exists and equals $M$. This proves the lemma.

## APPENDIX B

### *Proof of Lemma 3*

For each $t$, (*iii*) implies (Ref. 8, pages 204, 205, 226, 227, 230) the existence throughout $\Gamma_0$ of the $F$-derivatives of all orders of the map $\eta(\cdot, t)\colon \Gamma_0 \subset C^n \to C^n$. In particular, each partial derivative (17) exists in $\Gamma_0$ for any $t \geq 0$.†

---

* For example, if $l = 2$, $L(\mathscr{B}, L(\mathscr{B}, \cdots, L(\mathscr{B}, \mathscr{B})\cdots)) = L(\mathscr{B}, L(\mathscr{B}, L(\mathscr{B}, \mathscr{B})))$.

† See Section 8.9 of Ref. 8.

Given any $s \in \Gamma$,

$$\eta[s(t), t] = \int_0^1 d\eta[\beta s(t), t] d\beta \cdot s(t), \qquad t \geq 0$$

in which $d\eta[\beta s(t), t]$ is the $F$-derivative of $\eta(\cdot, t)$ at the point $\beta s(t)$ (i.e., $d\eta[\beta s(t), t]$ is the $n \times n$ matrix whose $ij$th element is $\partial\eta_i(z, t)/\partial z_j$ evaluated at $z = \beta s(t)$). By $(iv)$, the elements of $d\eta[\beta s(t), t]$ are bounded on $(\beta, t) \in [0, 1] \times [0, \infty)$. Thus, using $(ii)$, $N$ maps $\Gamma$ into $L_\infty(\mathrm{C})$.

Similarly, for any $s \in \Gamma$ and any $h \in L_\infty(\mathrm{C})$ such that $(s + h) \in \Gamma$,

$$\eta[s(t) + h(t), t] - \eta[s(t), t] - d\eta[s(t), t]h(t) = \int_0^1 \{d\eta[\beta(s(t) + h(t)) + (1 - \beta)s(t), t] - d\eta[s(t), t]\} d\beta \cdot h(t), \qquad t \geq 0.$$

This, together with the continuity described in $(iv)$, yields

$$\sup_{t \geq 0} |\eta[s(t) + h(t), t] - \eta[s(t), t] - d\eta[s(t), t]h(t)| = o(\|h\|) \qquad (22)$$

as $\|h\| \to 0$. Since the pointwise limit function of a sequence of (Lebesgue) measurable functions is measurable, and, for each $i = 1, 2, \cdots, n$, (22) holds with $h(t) = \sigma u(i)$ for $t \geq 0$, in which $\sigma$ is a scalar and $u(i)$ is the element of $\mathrm{C}^n$ with $u(i)_i = 1$ and $u(i)_j = 0$ for $i \neq j$, it easily follows that the elements of $d\eta[s(\cdot), \cdot]$ are measurable on $[0, \infty)$. By $(iv)$ these elements are bounded. Thus, using (22), $dN(s)$ exists and

$$[dN(s)h(t)]_i = \sum_{j_1=1}^{n} \frac{\partial\eta_i[s_1(t), \cdots, s_n(t), t]}{\partial z_{j_1}} h_{j_1}(t), \qquad t \geq 0$$

for each $i$. This shows that the $m = 1$ part of the lemma is true.

Now assume that the assertions of the lemma are true for $1 \leq m \leq l$, and again let $s \in \Gamma$ be given, and let $h \in L_\infty(\mathrm{C})$ satisfy $(s + h) \in \Gamma$.

By $(iv)$, each

$$\frac{\partial^{(l+1)}\eta_i[s_1(\cdot), \cdots, s_n(\cdot), \cdot]}{\partial z_{j_{(l+1)}} \cdots \partial z_{j_1}} \qquad (23)$$

is bounded on $[0, \infty)$. To see that each is measurable, observe that for $h_1, h_2, \cdots, h_l$ belonging to $L_\infty(\mathrm{C})$,

$$\sup_{t \geq 0} \max_i \Bigg| \sum_{j_1=1}^{n} \cdots \sum_{j_l=1}^{n} \frac{\partial^l\eta_i[s_1(t) + h_1(t), \cdots, s_n(t) + h_n(t), t]}{\partial z_{j_l} \cdots \partial z_{j_1}} h_{1j_1}(t) \cdots h_{lj_l}(t) - \sum_{j_1=1}^{n} \cdots \sum_{j_l=1}^{n} \frac{\partial^l\eta_i[s_1(t), \cdots, s_n(t), t]}{\partial z_{j_l} \cdots \partial z_{j_1}} h_{1j_1}(t)$$

$$\cdots h_{lj_l}(t) - \sum_{j_1=1}^{n} \cdots \sum_{j_{(l+1)}=1}^{n} \frac{\partial^{(l+1)}\eta_i[s_1(t), \cdots, s_n(t), t]}{\partial z_{j_{(l+1)}} \cdots \partial z_{j_1}} h_{1j_1}(t)$$

$$\cdots h_{lj_l}(t) h_{j_{(l+1)}}(t) | \leq o(\|h\|) \prod_{j=1}^{l} \|h_j\|, \tag{24}$$

which is a consequence of $(iv)$ and the relation

$$\frac{\partial^l \eta_i[s_1(t) + h_1(t), \cdots, s_n(t) + h_n(t), t]}{\partial z_{j_l} \cdots \partial z_{j_1}} - \frac{\partial^l \eta_i[s_1(t), \cdots, s_n(t), t]}{\partial z_{j_l} \cdots \partial z_{j_1}}$$

$$= \sum_{j_{(l+1)}=1}^{n} \frac{\partial^{(l+1)}\eta_i[s_1(t), \cdots, s_n(t), t]}{\partial z_{j_{(l+1)}} \partial z_{j_l} \cdots \partial z_{j_1}} h_{j_{(l+1)}}(t)$$

$$+ \int_0^1 \left\{ \sum_{j_{(l+1)}=1}^{n} \frac{\partial^{(l+1)}\eta_i[s_1(t) + \beta h_1(t), \cdots, s_n(t) + \beta h_n(t), t]}{\partial z_{j_{(l+1)}} \partial z_{j_l} \cdots \partial z_{j_1}} h_{j_{(l+1)}}(t) \right.$$

$$\left. - \sum_{j_{(l+1)}=1}^{n} \frac{\partial^{(l+1)}\eta_i[s_1(t), \cdots, s_n(t), t]}{\partial z_{j_{(l+1)}} \partial z_{j_l} \cdots \partial z_{j_1}} h_{j_{(l+1)}}(t) \right\} d\beta, \qquad t \geq 0.$$

It easily follows from (24) that each function (23) is the pointwise limit of a sequence of measurable functions, and is therefore measurable.

In particular, $\tilde{Q}_l(s)$ defined by

$$[\tilde{Q}_l(s)(p_1, \cdots, p_{(l+1)})(t)]_i$$

$$= \sum_{j_{(l+1)}=1}^{n} \sum_{j_l=1}^{n} \cdots \sum_{j_1=1}^{n} \frac{\partial^{(l+1)}\eta_i[s_1(t), \cdots, s_n(t), t]}{\partial z_{j_{(l+1)}} \partial z_{j_{(l)}} \cdots \partial z_{j_1}}$$

$$\cdot p_{1j_1}(t) p_{2j_2}(t) \cdots p_{(l+1)j_{(l+1)}}(t), \qquad t \geq 0$$

for $p_1, p_2, \cdots, p_{(l+1)}$ in $L_\infty(\mathrm{C})$ and $i = 1, 2, \cdots, n$, is a continuous multilinear mapping of $L_\infty(\mathrm{C})^{(l+1)}$ into $L_\infty(\mathrm{C})$.

Proceeding as in the proof of Lemma 2, let $Q_l(s)$ denote the usual associate of $\tilde{Q}_l(s)$ that belongs to $L(L_\infty(\mathrm{C}), L(L_\infty(\mathrm{C}), \cdots, L(L_\infty(\mathrm{C}), L_\infty(\mathrm{C})) \cdots))$ with $(l + 1)$ $L$'s, in which $L(A_1, A_2)$ stands for the set of continuous linear operators from the Banach space $A_1$ into the Banach space $A_2$. Using $\|d^lN(s + h) - d^lN(s) - Q_l(s)h\| = \sup\{\|d^lN(s + h)h_1 \cdots h_l - d^lN(s)h_1 \cdots h_l - Q_l(s)hh_1 \cdots h_l\|: \|h_1\| = \|h_2\| = \cdots = \|h_l\| = 1\}$, as well as our induction hypothesis and (24), we see that $\|d^lN(s + h) - d^lN(s) - Q_l(s)h\| = o(\|h\|)$ as $\|h\| \to 0$. Therefore $d^{(l+1)}N(s)$ exists and is equal to $Q_l(s)$. This completes the proof.

## APPENDIX C

### *Proof of Lemmas 4 and 5*

It suffices to prove the lemmas for $n = 1$ and $u(\tau_1, \cdots, \tau_l) = 1$ for $(\tau_1, \cdots, \tau_l) \in [0, \infty)^l$, and attention is now restricted to that case.

For $t > 0$, one has

$$\int_{[0,t]^l} \left| \int_0^t s(t, \tau)\tilde{h}(\tau, \tau_1, \cdots, \tau_l)d\tau \right| d(\tau_1, \cdots, d\tau_l)$$

$$\leq \int_{[0,t]^l} \int_0^t |s(t, \tau)| \cdot |\tilde{h}(\tau, \tau_1, \cdots, \tau_l)| d\tau d(\tau_1, \cdots, \tau_l)$$

$$\leq \int_0^t \int_{[0,\, t]^l} |\tilde{h}(\tau, \tau_1, \cdots, \tau_l)| d(\tau_1, \cdots, \tau_l) | s(t, \tau) | d\tau$$

$$\leq \sup_{\tau \geq 0} \int_{[0,\tau]^l} |h(\tau, \tau_1, \cdots, \tau_l)| d(\tau_1, \cdots, \tau_l) \cdot \sup_{t \geq 0} \int_0^t |s(t, \tau)| d\tau,$$

in which the measurability of

$$\int_0^t s(t, \tau)\tilde{h}(\tau, \tau_1, \cdots, \tau_l)d\tau \tag{25}$$

in $(\tau, \cdots, \tau_l)$, and the justification for the interchange of the order of integration, follow from theorems of Fubini and Tonelli (Ref. 12, pages 137–145). The measurability of (25) in $(t, \tau_1, \cdots, \tau_l)$ is also a consequence of these theorems.* Thus, since it is clear that $k$ is bounded, $(i)$ holds.

Now let $h$ and $s$ satisfy the conditions of part $(ii)$. Let $(t, \tau_1, \cdots, \tau_l) \in R_0(l)$ be given, let $\alpha, \alpha_1, \cdots, \alpha_l$ be real variables such that $(t + \alpha, \tau_1 + \alpha_1, \cdots, \tau_l + \alpha_l) \in R_0(l)$, and notice that

$$k(t + \alpha, \tau_1 + \alpha_1, \cdots, \tau_l + \alpha_l) - k(t, \tau_1, \cdots, \tau_l)$$

$$= \int_0^\infty [\tilde{s}(t + \alpha, \tau) - \tilde{s}(t, \tau)]\tilde{h}(\tau, \tau_1 + \alpha_1, \cdots, \tau_l + \alpha_l)d\tau$$

$$+ \int_0^\infty \tilde{s}(t, \tau)[\tilde{h}(\tau, \tau_1 + \alpha_1, \cdots, \tau_l + \alpha_l) - \tilde{h}(\tau, \tau_1, \cdots, \tau_l)]d\tau. \tag{26}$$

Using the hypothesis of part $(ii)$ concerning $s$, the boundedness of $h$ and $s$, and the uniform continuity of $h$ on compact subsets of $R_0(l)$, we see that each integral in (26) approaches zero as $(t + \alpha, \tau_1 + \alpha_1, \cdots, \tau_l + \alpha_l) \to (t, \tau_1, \cdots, \tau_l)$, showing that $(ii)$ is true.

Straightforward modifications of the proof of part $(iii)$ of Lemma 3 in Ref. 1 establish that $(iii)$ here holds.

With regard to part $(iv)$, using the theorems of Fubini and Tonelli cited above, and the proposition that a bounded measurable function

* Consider, for arbitrary finite $T > 0$, the existence and iterated-integral representations of the multiple integral $\int_{[0,\, T]^{(l+2)}} \tilde{s}(t, \tau)\tilde{h}(\tau, \tau_1, \cdots, \tau_l)d(t, \tau_1, \cdots, \tau_l, \tau)$.

on a set $E$ of finite measure is summable over $E$, we have

$$\int_0^t s(t, \tau) \int_0^\tau \cdots \int_0^\tau h(\tau, \tau_1, \cdots, \tau_l) d\tau_1 \cdots d\tau_l d\tau$$

$$= \int_0^t s(t, \tau) \int_{[0,\tau]^l} h(\tau, \tau_1, \cdots, \tau_l) d(\tau_1, \cdots, \tau_l) d\tau$$

$$= \int_0^t s(t, \tau) \int_{[0,t]^l} \tilde{h}(\tau, \tau_1, \cdots, \tau_l) d(\tau_1, \cdots, \tau_l) d\tau$$

$$= \int_{[0,t]^l} \int_0^t s(t, \tau) \tilde{h}(\tau, \tau_1, \cdots, \tau_l) d\tau d(\tau_1, \cdots, \tau_l)$$

$$= \int_0^t \cdots \int_0^t k(t, \tau_1, \ldots, \tau_l) d\tau_1 \cdots d\tau_l$$

for $t \geq 0$, which establishes $(iv)$ and completes the proof of Lemma 4.

Under the hypothesis of Lemma 5,

$$\int_{[0,t]^{(p+q)}} |h(t, \tau_1, \cdots, \tau_p) k(t, \tau_{(p+1)}, \cdots, \tau_{(p+q)})| d(\tau_1, \cdots, \tau_{(p+q)})$$

$$\leq \sup_{t \geq 0} \int_{[0,t]^p} |h(t, \tau_1, \cdots, \tau_p)| d(\tau_1, \cdots, \tau_p)$$

$$\times \sup_{t \geq 0} \int_{[0,t]^q} |k(t, \tau_{(p+1)}, \cdots, \tau_{(p+q)}| d(\tau_{(p+1)}, \cdots, \tau_{(p+q)})$$

for every $t \geq 0$, which proves the lemma.

## APPENDIX D

### *Proof of Lemma 6*

By the proof of Theorems 2.3 and 2.5 of Ref. 13, there exists a measurable function $\kappa$ from $R_0(1)$ into the set of complex $n \times n$ matrices such that the elements of $\kappa$ are bounded on bounded subsets of $R_0(1)$, and $\kappa$ satisfies the resolvent equations

$$\kappa(t, \tau) + \lambda(t, \tau) = \int_\tau^t \lambda(t, u) \kappa(u, \tau) du \tag{27}$$

$$\kappa(t, \tau) + \lambda(t, \tau) = \int_\tau^t \kappa(t, u) \lambda(u, \tau) du \tag{28}$$

for $t \geq \tau \geq 0$.

For each $p \in H(\mathrm{C})$, the function $q$ defined on $[0, \infty)$ by

$$q(t) = p(t) - \int_0^t \kappa(t, \tau)p(\tau)d\tau, \qquad t \geq 0 \tag{29}$$

belongs to $H(\mathrm{C})$, and, using (27) as well as theorems of Fubini and Tonelli (Ref. 12, pages 137–145) to justify an interchange of order of integration, it is simple matter to show that $q$ given by (29) satisfies (20) for each $p \in H(\mathrm{C})$. Similarly, it is essentially well known that (28) can be used to show that if there is a $q \in H(\mathrm{C})$ that satisfies (20) for a given $p \in H(\mathrm{C})$, then $q$ satisfies (29), which completes the proof.

## APPENDIX E

### *Proof of Lemma 7*

By Lemma 6 and its proof, $(I - \Lambda)$ is an invertible map of $L_\infty(\mathrm{C})$ onto $L_\infty(\mathrm{C})$, and there is a measurable matrix-valued $\kappa$, defined on $R_0(1)$ such that the elements of $\kappa$ are bounded on bounded subsets of $R_0(1)$, with the property that (28) is satisfied and

$$(I - \Lambda)^{-1}p(t) = p(t) - \int_0^t \kappa(t, \tau)p(\tau)d\tau, \qquad t \geq 0$$

for each $p \in L_\infty(\mathrm{C})$. Since $(I - \Lambda)^{-1}$ maps $L_\infty(\mathrm{C})$ into itself, it follows (Refs. 14 and 15) that each $\kappa_{ij}$ satisfies

$$\sup_{t \geq 0} \int_0^t |\kappa_{ij}(t, \tau)|d\tau < \infty. \tag{30}$$

Using (28), (30), and the boundedness of $\lambda$, we see that $\kappa$ is bounded on $R_0(l)$. Therefore, $\kappa \in S_n^{(1)}$.

Assume now that $\lambda$ satisfies the condition on $s$ of part (*ii*) of Lemma 4, recall that $\kappa$ satisfies (27), and let $r$ be defined by

$$r(t, \tau) = \int_0^t \lambda(t, u)\tilde{\kappa}(u, \tau)du$$

for $t \geq \tau \geq 0$.*

Let $t \geq 0$ be given. For arbitrary $i$ and $j$, let

$$\Delta_{ij}(\alpha, t) = \int_0^\infty |\tilde{r}_{ij}(t + \alpha, \tau) - \tilde{r}_{ij}(t, \tau)|d\tau$$

for $(t + \alpha) \geq 0$ (see the definition preceding Lemma 4 for the meaning of $\tilde{r}$; $r$ belongs to $S_n^{(1)}$ because $\kappa$ and $\lambda$ do and (27) is met). Notice that to complete the proof of our lemma, it suffices to show that $\Delta_{ij}(\alpha, t) \to 0$ as $\alpha \to 0$.

---

* With regard to the meaning of $\tilde{\kappa}$, see the definition immediately preceding Lemma 4.

It is clear that $\Delta_{ij}(\alpha, 0) \to 0$ as $\alpha \to 0$. Assume now that $t > 0$, and let $\alpha$ be such that $t - |\alpha| > 0$. We have

$$\Delta_{ij}(\alpha, t) = \int_0^{t-|\alpha|} |r_{ij}(t+\alpha, \tau) - r_{ij}(t, \tau)|d\tau + \int_{t-|\alpha|}^{t+|\alpha|} |\tilde{r}_{ij}(t+\alpha, \tau) - \tilde{r}_{ij}(t, \tau)|d\tau, \quad (31)$$

in which (by the boundedness of $r$) the second integral goes to zero as $\alpha \to 0$. Further,

$$\int_0^{t-|\alpha|} |r_{ij}(t+\alpha, \tau) - r_{ij}(t, \tau)|d\tau$$

$$= \int_0^{t-|\alpha|} \left| \sum_{k=1}^{n} \int_0^{t+\alpha} \lambda_{ik}(t+\alpha, u)\tilde{\kappa}_{kj}(u, \tau)du - \sum_{k=1}^{n} \int_0^{t} \lambda_{ik}(t, u)\, \tilde{\kappa}_{kj}(u,\tau)du \right| d\tau$$

$$\leq \sum_{k=1}^{n} \int_0^{t-|\alpha|} \int_0^{\infty} |\tilde{\lambda}_{ik}(t+\alpha, u) - \tilde{\lambda}_{ik}(t, u)| \cdot |\tilde{\kappa}_{kj}(u, \tau)|du d\tau$$

which, using the boundedness of the $\kappa_{kj}$, shows that the first integral on the right side of (31) also approaches zero as $\alpha \to 0$.

## APPENDIX F

### *Volterra Expansions on a Finite Time Interval*

In this appendix, $T$ denotes an arbitrary positive constant, $L_\infty$ (C) ($T$) stands for the complex Banach space of measurable complex column $n$-vector-valued functions $v$ defined on $[0, T]$ such that the $j$th component $v_j$ of $v$ satisfies $\sup_{t\in[0,T]} |v_j(t)| < \infty$ for $j = 1, 2, \cdots, n$, and where the norm $\|\cdot\|_T$ on $L_\infty(\mathrm{C})(T)$ is given by $\|v\|_T = \max_j \sup_t |v_j(t)|$, and for each $l = 1, 2, \cdots, R_0(l)(T)$ denotes the subset of $\mathrm{R}^{(l+1)}$ given by $R_0(l)(T) = \{(v_0, v_1, \cdots, v_l) \in \mathrm{R}^{(l+1)}: T \geq v_0 \geq v_i \geq 0$ for $i = 1, 2, \cdots, l\}$.

Similarly, for any positive integers $q$ and $l, S_q^{(l)}(T)$ denotes the set of complex $n \times q$ matrix-valued functions $h$ defined on $R_0(l)(T)$ such that each $h_{ij}$ is Lebesgue measurable and bounded on $R_0(l)(T)$.

We shall refer to the following two hypotheses.

$D$.1: There are elements $a$, $b$, $c$, and $d$ of $S_n^{(1)}(T)$ such that for each $p \in L_\infty(\mathrm{C})(T)$,

$$(Ap)(t) = \int_0^t a(t, \tau)p(\tau)d\tau$$

$$(Bp)(t) = \int_0^t b(t, \tau)p(\tau)d\tau$$

$$(Cp)(t) = \int_0^t c(t, \tau)p(\tau)d\tau$$

$$(Dp)(t) = \int_0^t d(t, \tau)p(\tau)d\tau$$

for $t \in [0, T]$.

*D*.2: With $\gamma$, $\Gamma_0$, $C^n$, and $\theta_c$ as indicated in the paragraph preceding B.2 of Section 2.4, $N$ is defined on $\Gamma = \{s \in L_\infty(C)(T): \| s \|_T < \gamma\}$ by

$$(Ns)(t) = \eta[s(t), t], \qquad t \in [0, T],$$

where $\eta$ is a map from $\Gamma_0 \times [0, T]$ into $C^n$ with the following properties:

(*i*) $\eta(\theta_c, t) = \theta_c$ for $t \in [0, T]$.

(*ii*) The function $\xi$ given by $\xi(t) = \eta[s(t), t]$, $0 \le t \le T$, is Lebesgue measurable on $[0, T]$ for each $s \in \Gamma$.

(*iii*) For each $t \in [0, T]$, $\eta(\cdot, t)$ is a continuous map of $\Gamma_0$ into $C^n$, and for each $t \in [0, T]$, for $1 \le i, j \le n$, and for any point $\alpha \in \Gamma_0$, the function $z_j \mapsto \eta_i(\alpha_1, \cdots, \alpha_{j-1}, z_j, \alpha_{j+1}, \cdots, \alpha_n, t)$ is differentiable with respect to the complex variable $z_j$ for $|z_j| < \gamma$. [This implies (Ref. 8, pages 204, 205, 226, 227, 230) the existence throughout $\Gamma_0$ of every $m$th order partial derivative

$$\frac{\partial^m \eta_i}{\partial z_{j_m} \partial z_{j_{m-1}} \cdots \partial z_{j1}} \tag{32}$$

for each $t$ and all $m$.]

(*iv*) For any $m, j_1, \cdots, j_m$, and $i$, the partial derivative (32), which we denote by $p(z_1, \cdots, z_n, t)$, satisfies the conditions that the function $t \mapsto p(0, \cdots, 0, t)$ is bounded on $[0, T]$, and that $p$ is uniformly continuous on closed subsets of $\Gamma_0$ uniformly in $t$, in the sense that given a closed $\Gamma_{00} \subset \Gamma_0$ and a $\delta_1 > 0$ there is a $\delta_2 > 0$ such that

$$| p(z_{a1}, \cdots, z_{an}, t) - p(z_{b1}, \cdots, z_{bn}, t) | \le \delta_1$$

for $t \in [0, T]$ whenever $z_a$ and $z_b$ are elements of $\Gamma_{00}$ such that $|z_a - z_b| < \delta_2$.

Direct modifications of the proof in Section 2.4.2 suffice to establish the following result, in which by Proposition 2′ we mean the corollary of Proposition 2 obtained from Proposition 2 by replacing $S_{n^l}^{(l)}$, $t \ge 0$,

$L_\infty(\mathrm{C})$, and $[0, \infty)$ by $S_{n^l}^{(l)}(T)$, $t \in [0, T]$, $L_\infty(\mathrm{C})(T)$, and $[0, T]$, respectively.*

*Theorem* 3: When D.1 and D.2 are met, there is a positive number $\delta$ and an open subset $S$ of $\Gamma$ of D.2 with the following properties.

(*i*) $S$ contains the origin in $L_\infty(\mathrm{C})(T)$, and for each $v \in L_\infty(\mathrm{C})(T)$ with $\|v\|_T < \delta$, there exist unique $x$, $y$, and $w$ of $S$, $L_\infty(\mathrm{C})(T)$, and $L_\infty(\mathrm{C})(T)$, respectively, such that (2), (3), and $y = Nx$ hold.

(*ii*) For each $l = 1, 2, \cdots$ there is a $k_l \in S_{n^l}^{(l)}(T)$ such that

$$w = \sum_{l=1}^{\infty} V_{k_l}(v) \quad \text{for} \quad \|v\|_T < \delta,$$

with the series uniformly convergent with respect to $\|v\|_T < \delta$, where $V_{k_l}(\cdot)$ is as indicated in Proposition 2′ (which is described just before Theorem 3).

(*iii*) Each $k_l$ can be taken to be continuous on $R_0(l)(T)$ when $a$, $b$, $c$, and $d$ are continuous on $R_0(1)(T)$.

## REFERENCES


1. I. W. Sandberg, "Expansions for Nonlinear Systems," B.S.T.J., this issue. See also I. W. Sandberg, "Functional Expansions for Nonlinear Systems," Proceedings of the 1981 Int. Conf. on Digital Signal Processing, Florence, September 2–5, 1981.
2. V. Volterra, "Sopra le Funzioni che Dipendono da Altre Funzioni, Nota 1," Rend. Lincei Ser. 4, 3 (1887), pp. 97–105.
3. V. Volterra, *The Theory of Functionals and of Integral and Integro-differential Equations*, New York: Dover, 1959.
4. M. Fréchet, "Sur les Fonctionnelles Continues," Ann. de L'Ecole Normale sup., 3rd Series, *27* (1910).
5. I. W. Sandberg, "On the Properties of Some Systems that Distort Signals — II," B.S.T.J., *43,* No. 1 (January 1964), pp. 91–112.
6. I. W. Sandberg, "On the Theory of Linear Multi-loop feedback Systems," in *Feedback Systems*, ed. J. B. Cruz, Jr., New York: McGraw-Hill, 1972.
7. J. G. Truxal, *Automatic Feedback Control System Synthesis*, New York: McGraw-Hill, 1955, p. 114.
8. J. Dieudonné, *Foundations of Modern Analysis*, New York: Academic Press, 1969.
9. A. Halme, J. Orava, and H. Blomberg, "Polynomial Operators in Nonlinear System Theory," Int. J. Syst. Sci. *2,* No. 1 (1971), pp. 25–47.
10. T. M. Flett, *Differential Analysis*, London: Cambridge University Press, 1980.
11. N. Wiener, *The Fourier Integral and Certain of its Applications*, New York: Dover Publications, 1933.
12. S. McShane, *Integration*, Princeton: Princeton Univ. Press, 1944.
13. R. K. Miller, *Nonlinear Volterra Integral Equations*, Menlo Park, California: Benjamin Inc., 1971.
14. D. C. Youla, "On the Stability of Linear Systems," IEEE Trans. Circ. Th., *CT-10,* No. 2 (1963), pp. 276–9.
15. C. A. Desoer and A. J. Thomasian, "A Note on Zero-State Stability of Linear Systems," Proc. First Annual Allerton Conf. Circ. and Syst. Th. (1963), pp. 50–2.


* In this connection, see the last paragraph of Section 2.3.2. Also, a hypothesis corresponding to B.3 is not needed because when D.1 and D.2 hold, and $L(t)$ is as indicated in Proposition 1 for $t \in [0, T]$, for each $p \in L_\infty(\mathrm{C})(T)$ there exists a unique $q \in L_\infty(\mathrm{C})(T)$ such that $p(t) = q(t) - \int_0^t c(t, \tau)L(\tau)q(\tau)d\tau$, $0 \le t \le T$. On the other hand, $\delta$ and $S$ of Theorem 3 depend on $T$.