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Error Probability of Partial-Response Continuous-Phase Modulation with Coherent MSK-Type Receiver, Diversity, and Slow Rayleigh Fading in Gaussian Noise

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This paper considers a class of constant-amplitude modulation schemes with good spectral main lobe and tail behavior. Detection is assumed to be coherent and the receiver is of offset-quadrature type, i.e., minimum shift keying (MSK) type, consisting of a linear filter in each quadrature arm followed by simple processing. Analytical error-probability formulas are derived for various modulation schemes and receiver filters for ideal diversity with maximal-ratio and selection combining. Independent slow Rayleigh fading in Gaussian noise is assumed and several numerical examples are given. Asymptotic behavior of the error probability for large signal-to-noise ratios is derived, and the relationship between the degree of smoothing in the partial-response continuous-phase modulation and the asymptotic error probability is shown for fading channels with and without diversity.

I. INTRODUCTION

The transmission of information over radio channels with multiple changing-propagation paths is subject to fading, i.e., random time variations of the receiver signal strength. For digital transmission over a fading channel, the time variations cause a varying error probability for all types of digital-modulation methods.

The application of bandwidth-efficient constant-amplitude modulation schemes to digital land-mobile radio has been considered in several papers.[1-5] Other investigations have shown that transmission

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over such channels is subject to fading.[6,7] In this paper we will consider slow Rayleigh fading where the density function for the signal-to-noise ratio (s/n) is

$$f(\gamma) = \frac{1}{\Gamma} e^{-\gamma/\Gamma}, \tag{1}$$

where Γ is the average signal-to-noise ratio. We define slow fading as the time-varying s/n γ, which is approximately constant over several transmitted bits (symbols). It will be seen below that the type of detectors considered for the modulation schemes in this paper operate over at most that number of symbols.

By combining a number of channels with independent Rayleigh fading, the density function for the resulting signal-to-noise ratio (1) can be improved. This is called diversity. Thus, with decreased probability of very low signal-to-noise ratios, the average error probability is improved by means of diversity.

This paper presents analytical, easy to use, bit-error probability formulas for smoothed continuous-phase constant-amplitude modulation with a simple coherent receiver and diversity. From these calculations we conclude that the increase in error probability owing to the degree of smoothing is smaller for a fading channel than for a nonfading channel. In Section IV we show an example of the difference between quadriphase shift keying (QPSK) and the schemes $3RC$ and $4RC$, which depict smoothed constant-amplitude modulations with narrow spectral main lobe and low spectral tails. The nonfading channel is approached with large numbers of diversity branches. The fading channel is approached with few branches of diversity. The gradual change from the two extreme cases is readily apparent.

1.1 *Probability of error*

Coherent transmission is considered. For quadriphase shift keying, 4-PSK (QPSK), there is a special case of the more general error-probability formula considered below. For QPSK and Binary Phase Shift Keying, 2-PSK (BPSK) the error probability for the Gaussian channel with ideal transmission and optimal detection is

$$P(\gamma) = Q(\sqrt{2\gamma}), \tag{2}$$

where $\gamma = E_b/N_o$ is the s/n (per information bit), E_b is the energy per information bit, and N_o is the (one-sided) spectral density for the additive white Gaussian noise. The function $Q(\cdot)$ is the error function associated with the normal distribution, i.e.,

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt. \tag{3}$$

For the family of modulation schemes considered in this paper, the error probability $P(\gamma)$ for the Gaussian channel is a linear sum of Q-function terms. We obtain the average bit-error probability for the fading case by averaging $P(\gamma)$ over the density function in eq. (1), i.e.,

$$P = \int_o^\infty f(\gamma)P(\gamma)d\gamma. \tag{4}$$

The averaging is performed with different density functions for the diversity cases below with different combiner strategies. The problem dealt with in this paper is finding the average bit-error probability (4) for the family of modulation schemes and diversity cases described briefly below.

The solution to the error-probability averaging in (4) is of interest in cellular systems with frequency reuse where cochannel interference is the main interference source and where diversity is used to improve the signal-to-noise density function.[3,6,7] When we consider so-called time-division retransmission schemes, it is realistic to consider space diversity with more than two diversity branches.[8–10]

1.2 Modulation schemes

Before we address the fading and diversity problems, we will give a short description of the modulation schemes considered in this paper. A family of binary constant-amplitude digital modulation schemes is defined by the transmitted signal

$$S(t, \underset{\sim}{\alpha}) = \sqrt{\frac{2E}{T}} \cos[2\pi f_o t + \phi(t, \underset{\sim}{\alpha})], \tag{5}$$

where $E = E_b$ is the symbol (bit-) energy, T is the symbol time, and f_o is the carrier frequency. The information-carrying phase is

$$\phi(t, \underset{\sim}{\alpha}) = 2\pi h \sum_{i=-\infty}^{\infty} \alpha_i q(t - iT), \tag{6}$$

where $\underset{\sim}{\alpha} = \cdots \alpha_{n-2}, \alpha_{n-1}, \alpha_n, \alpha_{n+1}, \alpha_{n+2} \ldots$ is a sequence of independent binary symbols $\alpha_i \in \{-1, +1\}$ (we will consider only binary schemes here) and h is the modulation index. The phase response is defined by

$$q(t) = \int_{-\infty}^{t} g(\tau)d\tau, \tag{7}$$

where $g(t)$ is a time-limited pulse defining instantaneous frequency.[11–13] The above family of schemes is considered in this paper only for modulation index $h = 1/2$. For this case, such modulation schemes as minimum shift keying (MSK)—or fast frequency shift keying (FFSK),[1] tamed-frequency modulation (TFM),[2] and Gaussian MSK (GMSK)[3,4]—are contained in the family (5). Different modulation schemes are obtained by changing the pulse shape $g(t)$. Thus, for MSK

$$g(t) = \begin{cases} 0 & t < 0, \quad t > T \\ \dfrac{1}{2T} & 0 \le t \le T, \end{cases} \tag{8}$$

and for TFM[2,12]

$$g(t) = \tfrac{1}{8}[g_o(t - T) + 2g_o(t) + g_o(t + T)], \tag{9}$$

where

$$g_o(t) = \frac{1}{T}\left[\frac{\sin\left(\dfrac{\pi t}{T}\right)}{\left(\dfrac{\pi t}{T}\right)} - \frac{\pi^2}{24} \cdot \frac{2\sin\left(\dfrac{\pi t}{T}\right) - \dfrac{\pi t}{T}\cos\left(\dfrac{\pi t}{T}\right) - \left(\dfrac{\pi t}{T}\right)^2 \sin\left(\dfrac{\pi t}{T}\right)}{\left(\dfrac{\pi t}{T}\right)^3}\right]. \tag{10}$$

The TFM pulse is infinite in time. Below we consider time-truncated versions of the pulse (9).

Raised-cosine pulses of various lengths LT[11–12] are also considered. For this case

$$g(t) = \begin{cases} \dfrac{1}{2LT}\left[1 - \cos\left(\dfrac{2\pi t}{T}\right)\right] & 0 \le t \le LT \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The notation LRC denotes a raised-cosine pulse of length LT or a modulation scheme (5), (6) based on the pulse LRC, shown in eq. (11). Additional details on schemes based on LRC can be found in Refs. 11, 12, and 13.

Duobinary MSK, i.e.,

$$g(t) = \begin{cases} \dfrac{1}{4T} & 0 \le t \le 2T \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

is also considered. This pulse is also denoted as $2REC$.[12]

Schemes based on pulses $g(t)$ of length larger than the symbol time T are so-called partial-response schemes. Controlled intersymbol interference is introduced. This improves the power spectra[12,14] and the detection efficiency under certain conditions.[12,15]

Figures 1 and 2 show the motivation for considering partial-response continuous-phase modulation schemes with overlapping pulses $g(t)$ of increasing length L. Figure 1 shows how the power spectral density (normalized to the total power) becomes more narrow for larger L values. The sidelobes are also low due to the large number of continuous derivatives in $g(t)$.[11,12,14] Figure 2 compares QPSK, MSK, and $3RC$. Note that the raised-cosine schemes have constant amplitude. By changing $g(t)$, further improvements of the spectral tails can be achieved.[2,12]

1.3 Detectors

We show in our references that the $h = 1/2$ schemes can be detected with good efficiency in an MSK detector[1] or an MSK-type detector (modified offset quadrature receiver),[13] which consists of a "matched" filter in each quadrature arm followed by very simple digital processing. In its most simple form, this processing consists of differential decoding in each quadrature arm[1,2] with differential encoding employed at the transmitter. For MSK, the receiver in Fig. 3a is optimum. MSK consists of linear modulation in each quadrature arm. A matched receiver filter whose impulse response is a half-cycle sinusoid of length $2T$ is the filter function $a_1(t)$ and $a_2(t)$ in this case.

An MSK receiver with modified filters was proposed in Ref. 2 for partial response schemes in the case of TFM. For the partial response case with overlapping pulses $g(t)$, the receiver shown in Fig. 3a is suboptimum. However, for $h = 1/2$ and for reasonably small L ($L \leqslant 4$), this receiver gives good results for low and intermediate signal-to-noise ratios.[2,13] The error probability shown in the graphs below is for binary decisions based on the filter output compared to a threshold which is 0.

Figures 3b and 3b show two examples of the receiver filter $a_1(t)$.[13] The receiver filter $a_2(t)$ is identical to $a_1(t)$. The first example is the so-called SPAM (selected pulse amplitude modulation) filter,[13] where we have used the same strategy in filter selection as was shown in Ref. 2. Note that the filter in Fig. 3b is truncated to $N_T = 4$ symbols. Figure 3c shows the so-called MIN (minimum energy signal) filter. It is shown in some detail in Ref. 13 that the preferable filter depends on the s/n . Filter SPAM is good for low s/n's, while filter MIN is good for high s/n's.[13] For $3RC$ the difference in performance is small, though.

Figure 4 shows the "detection eye," i.e., the input $\cos\{\phi(t, \underset{\sim}{\alpha})\}$ to the filter $a_1(t)$ for the $3RC$ scheme. (For further details see Ref. 13). The eye in the other quadrature arm is the same, but offset one symbol interval T.

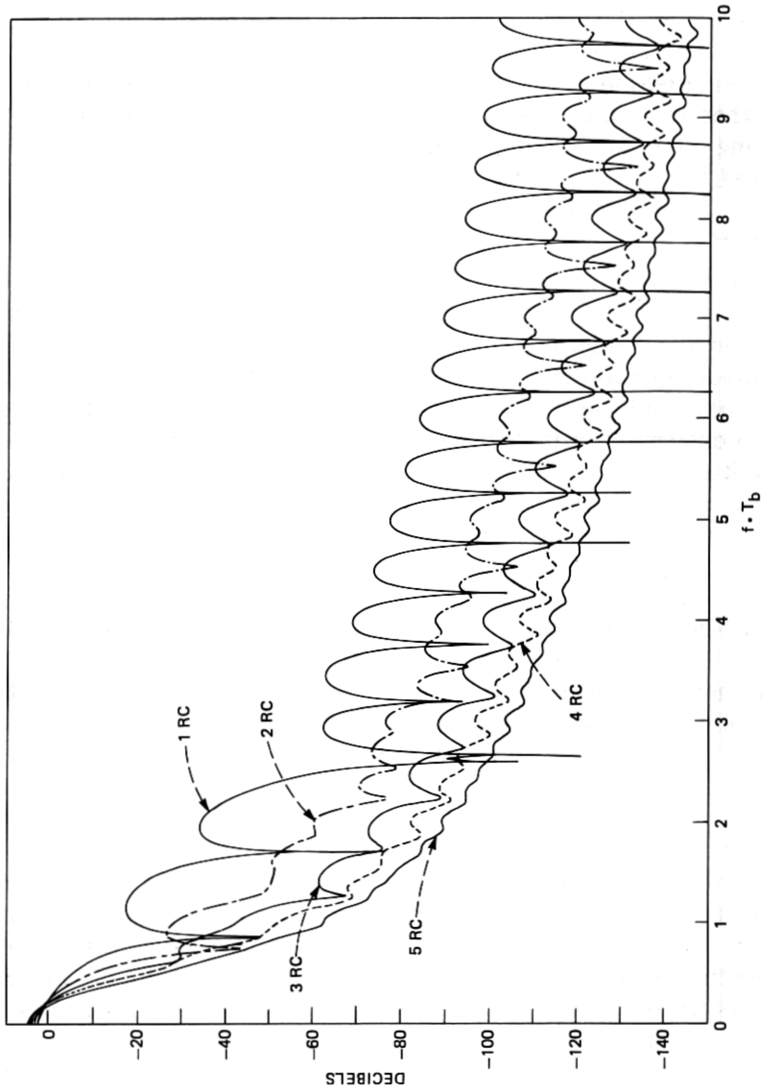


Fig. 1—Power spectra for binary RC schemes when $h = 1/2$ and $T_b = T$.

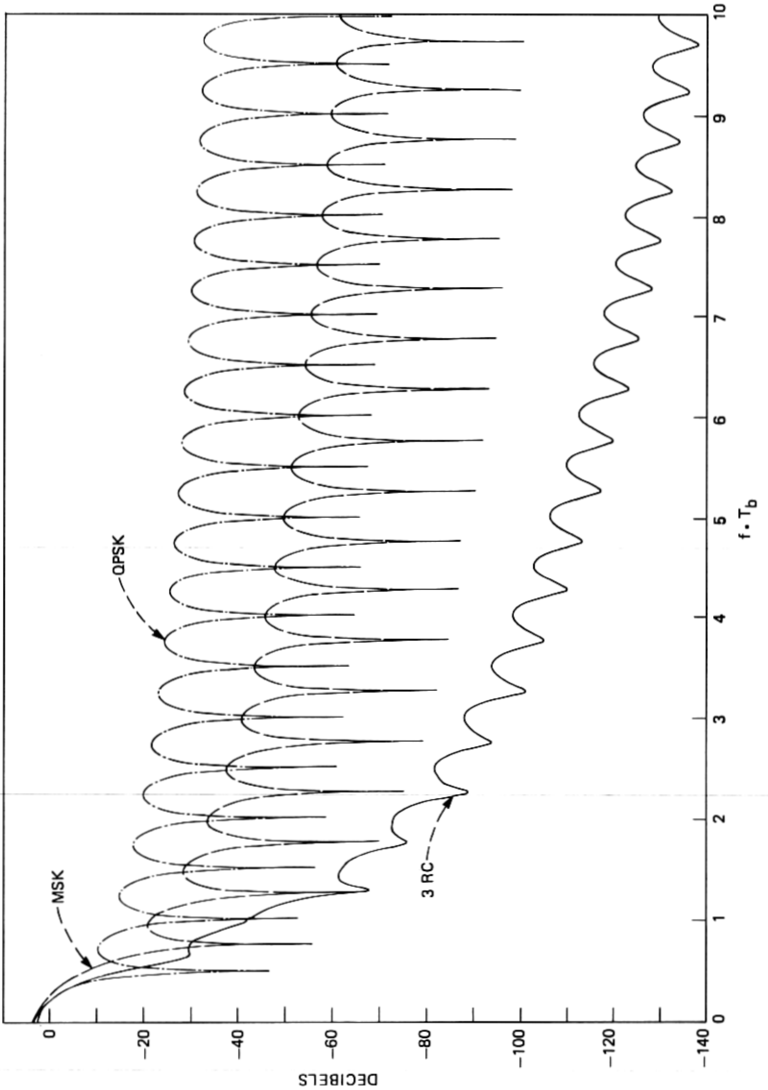
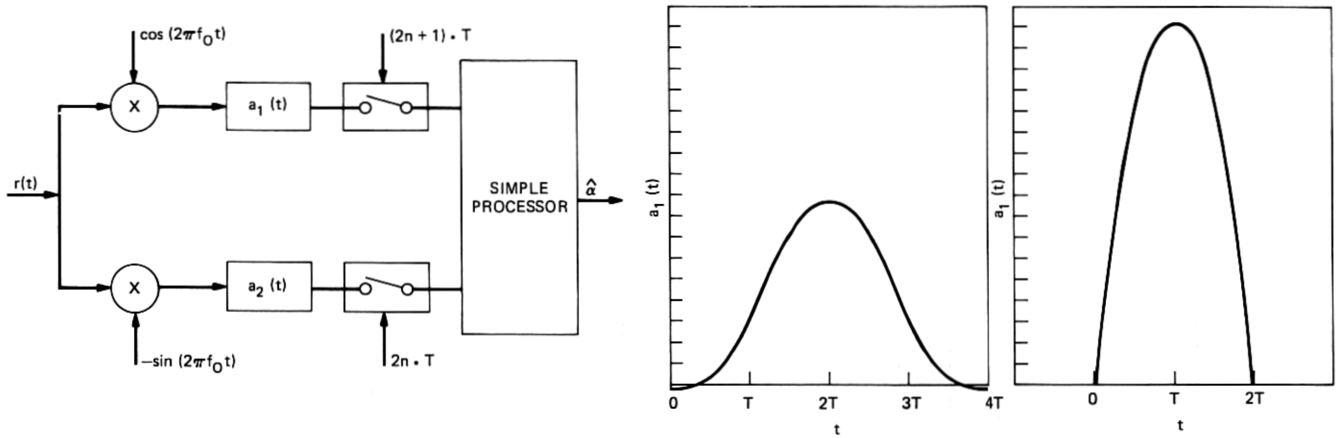


Fig. 2—Power spectra for binary $3RC$ with $h = 1/2$, QPSK, and MSK.



(a) (b) (c)

Fig. 3a—Receiver structure for the modified offset quadrature receivers for partial-response CPM.

Fig. 3b—The SPAM filter for a binary, $h = 1/2$, $3RC$ receiver.

Fig. 3c—The MIN filter for a binary, $h = 1/2$, $3RC$ receiver.

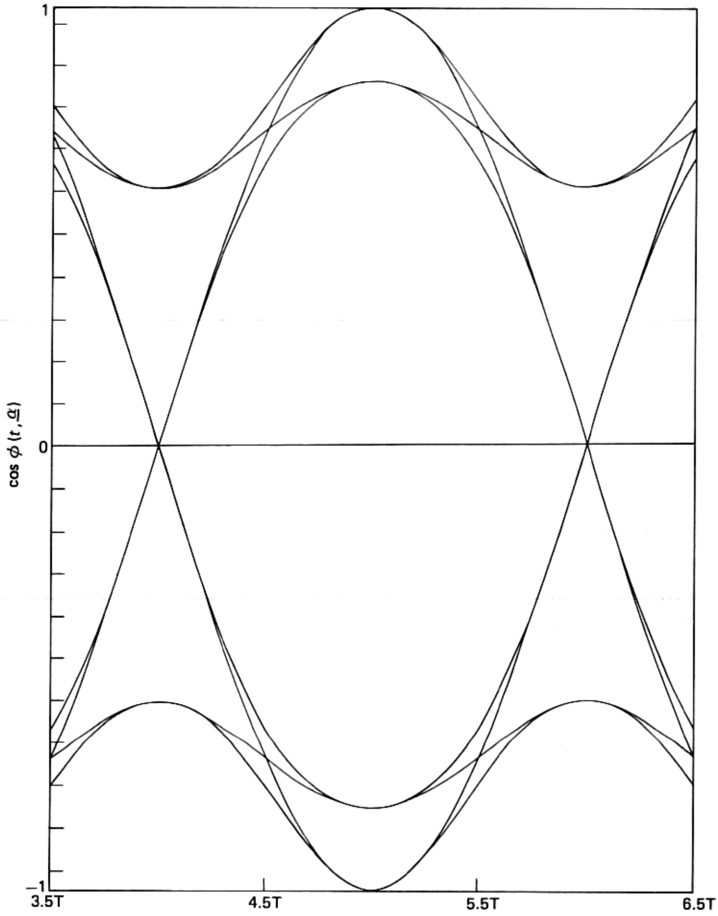


Fig. 4—Cos $\phi(t, \underline{\alpha})$ eye pattern of a binary $3RC$, $h = 1/2$, scheme. Note how open the eye is at the detection point ($t = 5T$ in this time scale).

1.4 Error probability in Gaussian noise

The average error probability P for a given modulation scheme—given pulse shape $g(t)$—and for a given receiver filter $a(t) = a_1(t) = a_2(t)$ can be evaluated exactly. For any given data sequence, the decision after the filter is a comparison between a Gaussian variable and a threshold which is zero in Fig. 3a. (For details see Refs. 13, 15, 16, 17, and 18.) It is shown in Ref. 13 that various filters are best at various levels of P (various values of s/n). Note that by error probability P we consistently refer to the average error probability associated with one binary decision after the receiver filter $a(t)$. Because of the phase ambiguity of π, the bit-error probability of the modulation scheme, P_{bit}, is affected by the resolution of this ambiguity.[1,2] With differential decoding we have[19]

$$P_{bit} = 2P(1 - P). \tag{13}$$

This will be discussed somewhat in Section V. Until then the calculations will be done using P (before the differential decoding).

In Ref. 13 we see that the average error probability P for the partial-response scheme with an MSK-type receiver is

$$P = \sum_{i=1}^{m} C_i Q\left(\sqrt{d_i^2 \frac{E_b}{N_o}}\right). \tag{14}$$

Here $E_b = E$, N_o is the spectral density of the one-sided Gaussian noise, $Q(\cdot)$ is the error function associated with the normal distribution, d_i^2 is the squared Euclidean distance associated with a signal corresponding to data sequence number i received in the fixed filter $a(t)$ (see Refs. 13 and 15 for details), and C_i is the probability of that specific signal. There are at most $m = 2^{N_T+L+1}$ different signals in (14), where N_T is the filter length in bit intervals (some of the distance values are the same for several data sequences due to symmetry. (See Ref. 13 for details.)

Assuming independent data symbols with $p(-1) = p(+1) = 1/2$, then

$$C_i = 1/m. \tag{15}$$

For QPSK and MSK we have $d^2 = 2$; thus,

$$P = Q\left(\sqrt{\frac{2E_b}{N_o}}\right) \tag{16}$$

with the optimum receiver filter.

For the general partial-response case, the sum (14) consists of several terms with several squared distance values d_i^2, where one of them is the minimum squared Euclidean distance. Methods for the calculation of d_i^2, $i = 1 \cdots m$, are given in Ref. 13. The parameters d_i^2 are independent of signal-to-noise ratio. They depend only on the data sequence, the pulse shape $g(t)$, and the receiver filter $a(t)$.

Figures 5 and 6 show the calculated average error probability P, using eq. (14), for a number of modulation schemes where the receiver filter is the SPAM filter.[13] The technique described in Ref. 2 was used for selecting the filter. Several different pulse shapes are considered (see Ref. 13 for details). For comparison, the error probability for QPSK is also shown. It is evident that the longer and smoother pulse $g(t)$, the better the power spectrum and the larger the penalty in E_b/N_o compared to QPSK. However, the penalty—using, for example, $3RC$—is only 0.5 dB at $P = 10^{-3}$. The penalty for TFM at $P = 10^{-3}$ is about 1.0 dB. For lower P values the penalty is larger.

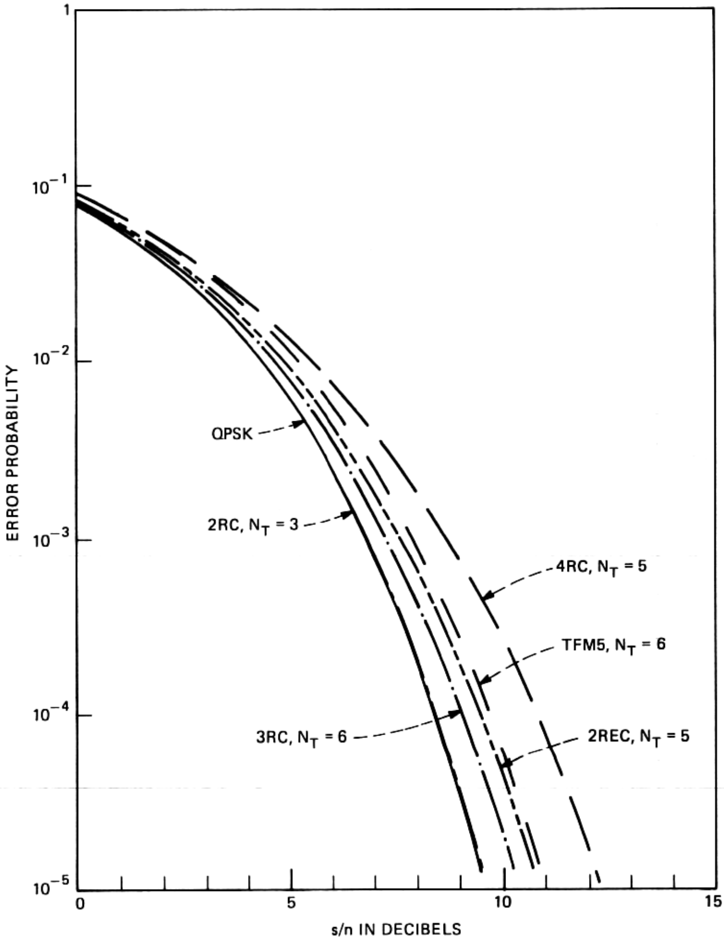


Fig. 5—Error probability in detecting the phase node sequence (output of the receiver filter) for SPAM receivers for various binary, $h = 1/2$, schemes.

The rest of the paper is organized as follows: In Section II we derive formulas for error probability for the partial-response continuous-phase modulation schemes with M branch diversity and maximal-ratio combining. Section III contains the corresponding results for selection combining, and Section IV presents some numerical results. Section V contains a discussion and conclusions.

II. MAXIMAL-RATIO-COMBINING

First we consider the problem of calculating the average error probability

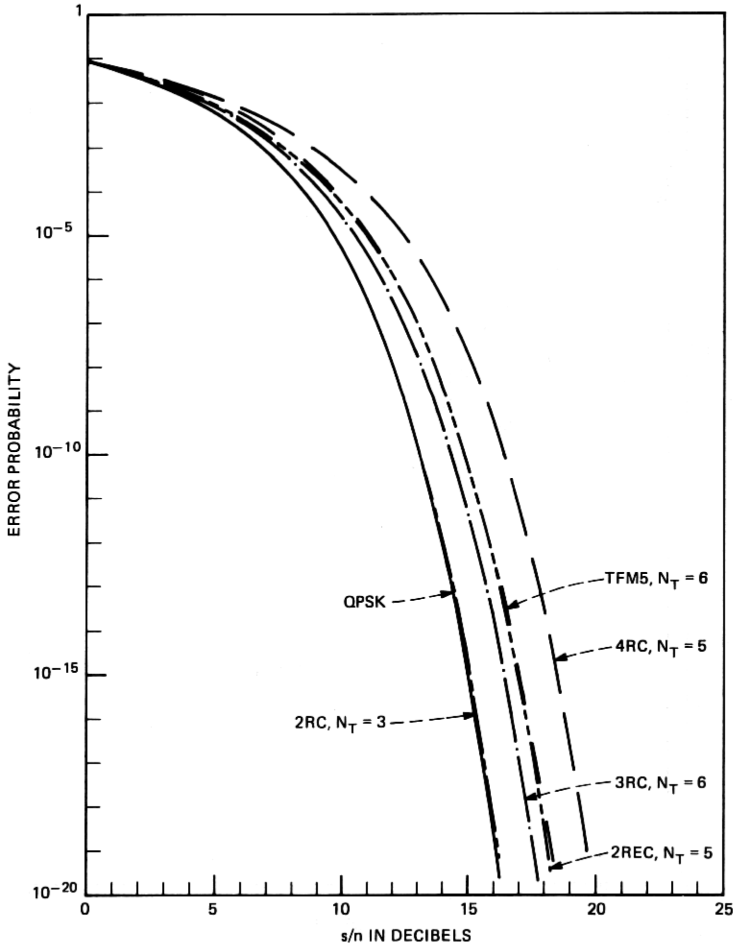


Fig. 6—Error probability in detecting the phase node sequence (output of the receiver filter) for SPAM receivers for various binary, $h = 1/2$, schemes.

$$P = \int_o^\infty f(\gamma)P(\gamma)d\gamma, \tag{17}$$

where $P(\gamma)$ is the average error probability of the modulation scheme (with coherent detection) for Gaussian noise at the s/n γ given by (14),

$$P(\gamma) = \sum_{i=1}^{m} C_i Q(\sqrt{d_i^2\gamma}), \tag{18}$$

and where $f(\gamma)$ is the density function for γ. Note that the receiver filter $a(t)$ operates over N_T symbols, typically smaller than 6. Slow fading is assumed and the averaging is performed over one "block" of length N_T, much the same way as is shown in Ref. 20.

The instantaneous s/n in diversity branch k is assumed to be γ_k, with equal average value for all branches. For Rayleigh fading, the γ_k have the probability density function of eq. (1).

Assume ideal maximal-ratio combining.[6,7] This combiner must know each path magnitude and phase to perform perfect combining and must have the property that the output signal-to-noise ratio is the sum of the instantaneous branch s/n's, i.e.,

$$\gamma = \sum_{k=1}^{M} \gamma_k. \tag{19}$$

The random variable γ has the density function[6,7]

$$f(\gamma) = \frac{1}{\Gamma}\left(\frac{\gamma}{\Gamma}\right)^{M-1} \frac{1}{(M-1)!}\, e^{-\gamma/\Gamma} \tag{20}$$

for M-branch maximal-ratio combining, assuming independent Rayleigh fading in each branch. The average receiver-output s/n is

$$E\{\gamma\} = M\Gamma. \tag{21}$$

References 21 and 22 show that the average bit-error probability for BPSK (QPSK) with coherent detection, maximal-ratio combining, independent Rayleigh fading, and Gaussian noise is

$$P = \frac{1}{2}\left\{1 - \sqrt{\frac{\Gamma}{1+\Gamma}}\left[1 + \frac{1}{1!2}(1+\Gamma)^{-1} + \frac{1\cdot 3}{2!2^2}(1+\Gamma)^{-2} + \cdots + \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2M-3)}{(M-1)!2^{M-1}}(1+\Gamma)^{-(M-1)}\right]\right\}, \tag{22}$$

where Γ is the average s/n per branch and where M is the number of branches, $M \geq 2$. Corresponding formulas are also described in Refs. 23 through 25. For $M = 1$, see Ref. 7.

The average error probability for a modulation scheme with bit-error probability $P(\gamma) = Q(\sqrt{2\alpha\gamma})$ for the additive white Gaussian channel is given by eq. (22) with Γ replaced by $\alpha\Gamma$ (see Refs. 21 and 22).

For the case of coherent MSK-type reception of partial-response continuous-phase modulated signals (see Section I), the error probability after the detection filter but before differential detection is a weighted sum of Q-functions, as shown in eq. (14). Thus, the bit-error probability for the fading and diversity case is given by

$$P = \sum_{i=1}^{m} \frac{C_i}{2} \left\{ 1 - \sqrt{\frac{\frac{d_i^2\Gamma}{2}}{1 + \frac{d_i^2\Gamma}{2}}} \right.$$

$$\cdot \left[1 + \frac{1}{1!2} \left(1 + \frac{d_i^2\Gamma}{2}\right)^{-1} + \frac{1\cdot 3}{2!2^2} \left(1 + \frac{d_i^2\Gamma}{2}\right)^{-2} \right.$$

$$\left.\left. + \cdots + \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2M-3)}{(M-1)!2^{(M-1)}} \left(1 + \frac{d_i^2\Gamma}{2}\right)^{-(M-1)} \right] \right\}, \qquad (23)$$

where C_i, d_i^2, $i = 1 \cdots m$ depend on the modulation scheme and the receiver filter used (see Section I). Numerical calculations of (23) for various modulation schemes and various M are presented in Section IV below.

For large s/n's Γ, the formula (22) reduces to[6,7,21]

$$P \approx \frac{(2M-1)!}{M!(M-1)!} \left(\frac{1}{4\Gamma}\right)^M. \qquad (24)$$

It is easy to adapt this formula to the partial-response continuous-phase modulation case. For large s/n's eq. (23) can be written as

$$P \approx \frac{(2M-1)!}{M!(M-1)!} \left(\frac{1}{4\Gamma}\right)^M \cdot \sum_{i=1}^{m} \frac{C_i}{\left(\frac{d_i^2}{2}\right)^M}. \qquad (25)$$

To evaluate the asymptotic s/n difference between different schemes, it is convenient to write eq. (25) as

$$P \approx \frac{(2M-1)!}{M!(M-1)!} \left\{ \frac{1}{4\Gamma \left[\frac{1}{\sum_{i=1}^{m} \frac{C_i}{\left(\frac{d_i^2}{2}\right)^M}} \right]^{\frac{1}{M}}} \right\}^M \qquad (26)$$

The special case of BPSK (2PSK) is given by $m = 1$, $C_i = 1$, $d_i^2 = 2$.

Thus, the asymptotic E_b/N_o difference between two schemes, e.g., between QPSK (BPSK) and some partial-response CPM schemes is given by the factor

$$\frac{1}{\left[\sum_{i=1}^{m} \frac{C_i}{\left(\frac{d_i^2}{2}\right)^M} \right]^{\frac{1}{M}}} \qquad (27)$$

From the relationships above and (17) we can draw the following conclusions:

For $M = 1$, the difference in E_b/N_o between a partial-response scheme and QPSK is smaller than the difference given by $d^2_{\min}$ alone. All the d_i^2 affect the difference with equal weight $C_i = 1/m$. For increasing M values it can easily be concluded from the average in eq. (27) that the smallest d_i^2, i.e., $d^2_{\min}$, will play an increasing role and dominate the difference for large M, just as for the Gaussian channel.

These relative performance differences for various M values are illustrated by the numerical examples in Section IV.

III. SELECTION COMBINING

In this section we will derive formulas corresponding to eqs. (22), (23), and (24) for the case of diversity with ideal selection combining.

The ideal selection-diversity combiner is defined here as a device that selects that diversity branch with the largest s/n for bit decisions. The same branch is used for all symbols over one time interval for the receiver filter, under the assumption of slow fading.

Let γ_k be the instantaneous signal-to-noise ratio in branch k with average value Γ equal for all branches. For Rayleigh fading, the γ_k have the probability density function of eq. (1). As shown in Refs. 6 and 7, the probability density function for the ideal selection-combiner output s/n γ is

$$f(\gamma) = \frac{M}{\Gamma} e^{-\gamma/\Gamma}(1 - e^{-\gamma/\Gamma})^{M-1}, \tag{28}$$

where Γ is the average s/n per branch and M is the number of branches. The average receiver-output s/n in this case is

$$E\{\gamma\} = \Gamma \cdot \sum_{k=1}^{M} \frac{1}{k}, \tag{29}$$

which of course increases more slowly with increasing M than the corresponding average for maximal ratio combining (21).[6,7]

First we derive an analytical solution to (17) for selection combining and coherent BPSK. It is then easy to apply this solution to the error probability for partial-response continuous-phase modulation (14). The probability density function (28) can be written as

$$f(\gamma) = \frac{M}{\Gamma} \sum_{j=0}^{M-1} (-1)^j \binom{M-1}{j} e^{-\frac{\gamma}{\Gamma}(1+j)}. \tag{30}$$

The average bit-error probability for BPSK, coherent-detection and M-branch diversity with selection combining is

$$\begin{aligned} P &= \int_o^\infty f(\gamma) Q(\sqrt{2\gamma}) d\gamma \\ &= \frac{M}{\Gamma} \sum_{j=o}^{M-1} (-1)^j \binom{M-1}{j} \int_o^\infty Q(\sqrt{2\gamma}) e^{-\frac{\gamma}{\Gamma}(1+j)} d\gamma. \end{aligned} \tag{31}$$

We evaluate (31) for the more general family of modulation schemes where the bit-error probability for the Gaussian channel is $P(\gamma) = Q(\sqrt{2\alpha\gamma})$; here α is a constant ≤ 1.

The integral above can be evaluated analytically as

$$\int_o^\infty Q(\sqrt{2\alpha\gamma})e^{-\gamma/\Gamma(1+j)}d\gamma$$

$$= \int_o^\infty \left\{ \int_{\sqrt{2\alpha\gamma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \right\} e^{-\frac{\gamma}{\Gamma}(1+j)} d\gamma$$

$$= \frac{1}{2} \cdot \frac{\Gamma}{1+j} \left(1 - \frac{1}{\sqrt{1 + \frac{1+j}{\alpha\Gamma}}} \right). \tag{32}$$

Thus, with (31) we have P:

$$P = \frac{M}{2} \sum_{j=0}^{M-1} \frac{(-1)^j}{(1+j)} \cdot \binom{M-1}{j} \left(1 - \frac{1}{\sqrt{1 + \frac{1+j}{\alpha\Gamma}}} \right), \tag{33a}$$

which can be written more compactly as

$$P = \frac{1}{2} \sum_{j=0}^{M} \frac{(-1)^j \binom{M}{j}}{\sqrt{1 + \frac{j}{\alpha\Gamma}}}. \tag{33b}$$

For the general case with an MSK-type receiver for a partial-response continuous-phase modulation scheme with $h = 1/2$, as shown in eq. (14), we arrive at the analytical error-probability formula

$$P = \sum_{i=1}^{m} C_i \cdot \frac{1}{2} \sum_{j=0}^{M} \frac{(-1)^j \binom{M}{j}}{\sqrt{1 + \frac{2j}{d_i^2\Gamma}}}, \tag{34}$$

where C_i, d_i^2, $i = 1 \cdots m$ are defined by the modulation scheme and by the receiver. Numerical examples for specific transmitted-signal formats, receiver filters, and number of diversity branches M are given for (34) in Section IV below.

Next, we evaluate the dominating term in (34) for large signal-to-noise ratios Γ. For simplicity we do this for BPSK ($\alpha = 1$). This can then easily be extended to the general case (34), as we will see. For large Γ's, eq. (33) with $\alpha = 1$ can be written as

$$P = \frac{1}{2} \sum_{i=0}^{M} (-1)^i \binom{M}{i}$$

$$\cdot \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2^k k!} \left(\frac{i}{\Gamma} \right)^k \right\}. \qquad (35)$$

Since

$$\sum_{i=0}^{M} (-1)^i \binom{M}{i} = 0 \qquad (36)$$

we have

$$P = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2^k \cdot k!} \left(\frac{1}{\Gamma} \right)^k \cdot \sum_{i=0}^{M} (-1)^i \binom{M}{i} i^k. \qquad (37)$$

The quantity

$$S = \sum_{i=0}^{M} (-1)^i \binom{M}{i} i^k \qquad (38)$$

is related to the Stirling numbers of the second kind (see Ref. 26, Chapter 24 and Ref. 27, p. 33). S is 0 for $k < M$ and

$$S = (-1)^M \cdot M! \qquad (39)$$

for $k = M$. S can also be evaluated for $k > M$, as shown in Refs. 26 and 27. Thus, for large Γ's, the error probability behaves like

$$P \cong \frac{1 \cdot 3 \cdot \cdots \cdot (2M-1)}{2^{M+1}} \frac{1}{\Gamma^M}. \qquad (40)$$

The power of $1/\Gamma$ is the same as for M-branch diversity with maximal-ratio combining. The multiplying coefficient is larger, however.

For high-average-branch s/n's Γ, the asymptotic error-probability behavior for coherent BPSK with M-branch diversity is shown in Table I below.

Table 1—Asymptotic behavior of average probability of error for large average per-branch signal-to-noise Γ for selection and maximal-ratio combining using *BPSK* (QPSK).

M	Selection Combining	Maximal-Ratio Combining
1	$\frac{1}{4\Gamma}$	$\frac{1}{4\Gamma}$
2	$\frac{3}{8}\frac{1}{\Gamma^2}$	$\frac{3}{16}\frac{1}{\Gamma^2}$
3	$\frac{15}{16}\frac{1}{\Gamma^3}$	$\frac{5}{32}\frac{1}{\Gamma^3}$
4	$\frac{105}{32}\frac{1}{\Gamma^4}$	$\frac{35}{256}\frac{1}{\Gamma^4}$

For partial response $h = 1/2$ continuous phase-modulation with an MSK-type receiver, the asymptotic behavior for selection combining is

$$P = \frac{1 \cdot 3 \cdot \cdots \cdot (2M-1)}{2^{(M+1)}} \left\{ \frac{1}{\Gamma \left[\dfrac{1}{\sum_{i=1}^{m} \dfrac{C_i}{\left(\dfrac{d_i^2}{2} \right)^M}} \right]^{\frac{1}{M}}} \right\}^M \tag{41}$$

Comparing formula (41) with the corresponding one for maximal-ratio combining (24), it can immediately be seen that the relative asymptotic performances of various modulation schemes are the same for the selection combiner as for the maximal-ratio combiner. This is also illustrated by the curves in Section IV.

IV. NUMERICAL RESULTS

Formula (23) for maximal-ratio combining and (34) for selection combining are used to calculate the error probability P in the figures below. Again note that the error probability is that found after the filter in the quadrature arm in Fig. 1.

Figure 7 shows the error probability P versus the average per-branch (and per-bit) signal-to-noise ratio Γ for the modulation scheme $3RC$, $h = ½$, with a receiver filter of MIN-type.[13] Formula (23) was used and QPSK results are shown for comparison. Note that the difference between QPSK and $3RC$ grows with M. Also compare the difference in Figs. 5 and 6 for the nonfading-additive white Gaussian channel. As was predicted by the formula in Section II above, the asymptotic difference between QPSK and $3RC$ is smaller for $M = 1$ than for $M = 16$ and for the additive white Gaussian channel. This is due to the fact that, for low M, low signal-to-noise ratios dominate the density function $f(\gamma)$, as shown in eq. (1). For this region of γ, the difference between the error probability for QPSK and $3RC$ is small for the additive white Gaussian channel (see Figs. 5 and 6).

Figure 8 shows the same modulation and diversity schemes as Fig. 7 with the exception that for $3RC$ a SPAM-receiver filter is used (see the introduction and Ref. 13). Notice that the differences between the corresponding curves in Figs. 7 and 8 are small. The MIN filter is better than the SPAM filter for high s/n's over the additive white Gaussian channel (see Figs. 5 and 6 and Ref. 13). The SPAM filter is better than the MIN filter for low s/n's. This explains the small but noticeable differences between the $3RC$ curves in Figs. 7 and 8.

Figures 9 and 10 show the $3RC$-MIN and $3RC$-SPAM cases for selection combining with QPSK results shown for comparison. Formula (34) was

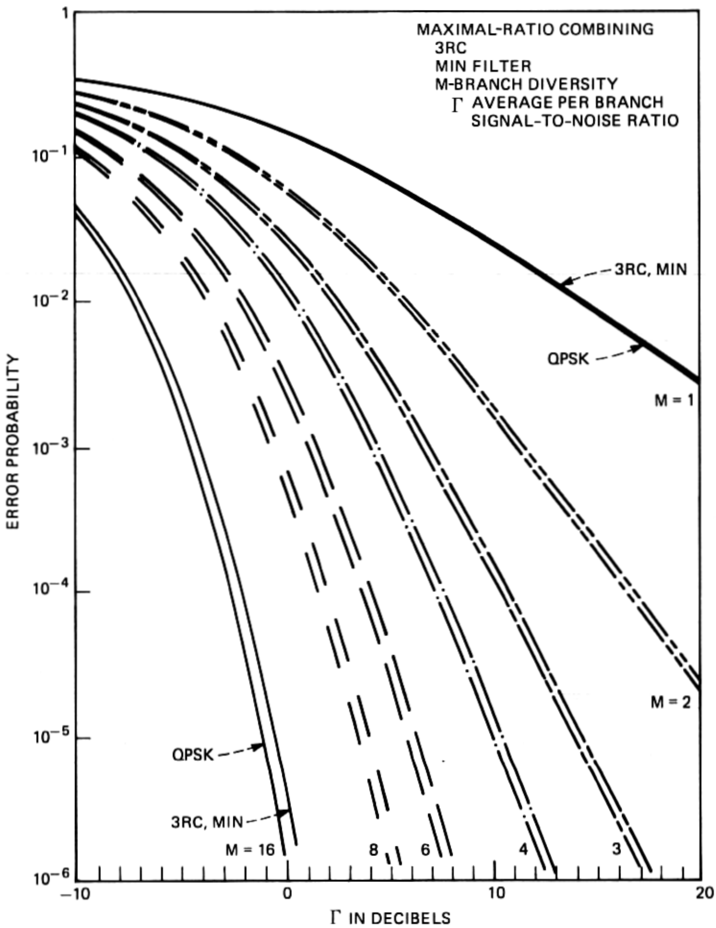


Fig. 7—Error probability P vs average-per-branch s/n Γ with $3RC$, $h = 1/2$, receiver with MIN-filter. (The curves are shown without differential encoding/decoding.)

used for the computations. Note that the same relative behavior is present in Figs. 9 and 10 as was shown in Figs. 7 and 8. Also note the larger per-branch s/n required for selection combining compared to maximal-ratio combining for equal M.[6,7]

The exact analytical formulas (23) and (34) are easy and straightforward to evaluate. For example, for $M = 16$ and for the $3RC$-SPAM case with a filter of length 4 binary symbols, $16x2^8$ terms are used in the sums. In general, $Mx2^{N_T+L+1}$ terms are summed where N_T is the filter impulse response length in bits.[13]

Figures 11 to 14 show the error probability P versus the average-per-branch s/n Γ for an increasing number of diversity branches with

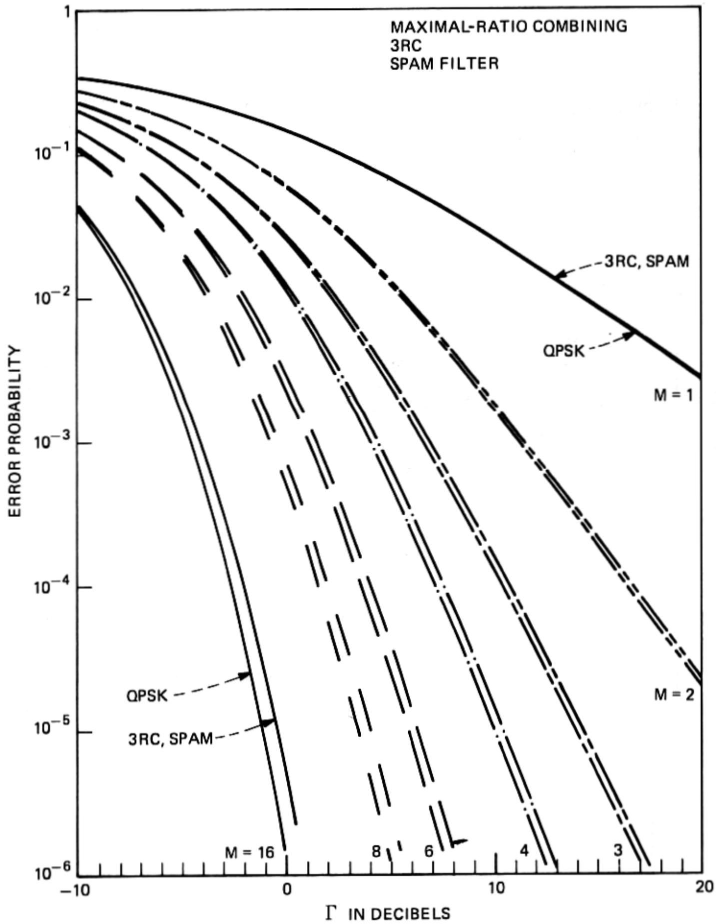


Fig. 8—Error probability P vs average-per-branch s/n Γ with $3RC$ and SPAM-filter M-branch diversity with maximal-ratio combining.

maximal-ratio combining. Again, QPSK (BPSK) results are shown for comparison. The same group of modulation schemes and receiver filters are used. Note the widening spread for increasing M, as expected. The corresponding case for selection combining is shown in Figs. 15 through 17 (and in Fig. 11 for the $M = 1$ case).

Figure 18 shows another way of presenting the relative error-probability results. This figure shows the required receiver output s/n $(M \cdot \Gamma)$ to achieve bit-error probability 10^{-3} as a function of the number of branches of diversity M. Maximal-ratio combining is assumed. The modulation schemes are coherent $2RC$, $3RC$, and $4RC$ with SPAM filters,[13] with QPSK shown for reference. For increasing M, the s/n for

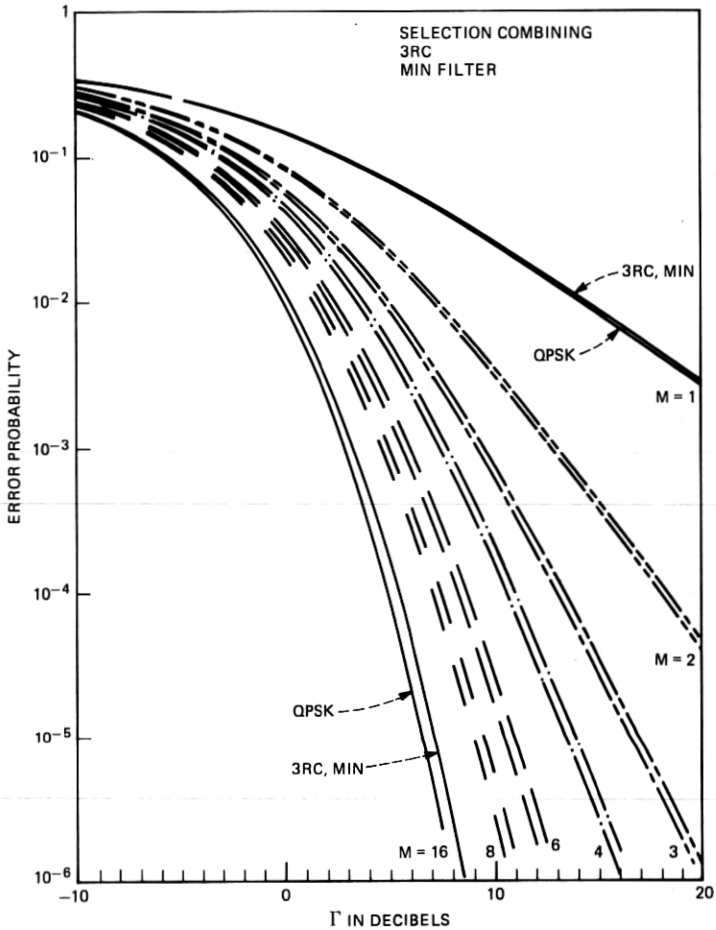


Fig. 9—3RC with MIN filter and M-branch diversity with selection combining.

the additive Gaussian channel is approached. Note the relative position of the curves for 2RC, 3RC, and 4RC. For further numerical results, see Ref. 28.

V. DISCUSSION AND CONCLUSIONS

The curves in Figs. 7 through 18 are calculated with the assumption that the receiver can resolve phase ambiguity of 180 degrees in the detection process.[1,2] This is normally done by employing differential encoding and decoding[1–3,17] and, likewise, for QPSK. For this case, the bit-error probability for the Gaussian channel is[19]

$$P_{bit} = 2P(1 - P), \tag{42}$$

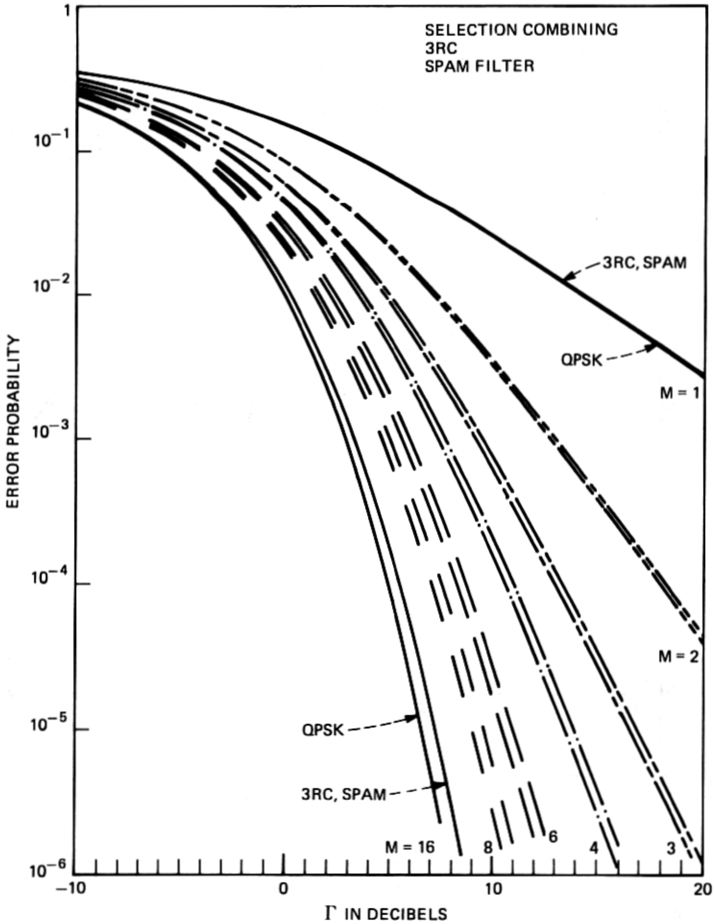


Fig. 10—3*RC* with SPAM-filter and *M*-branch diversity with selection combining.

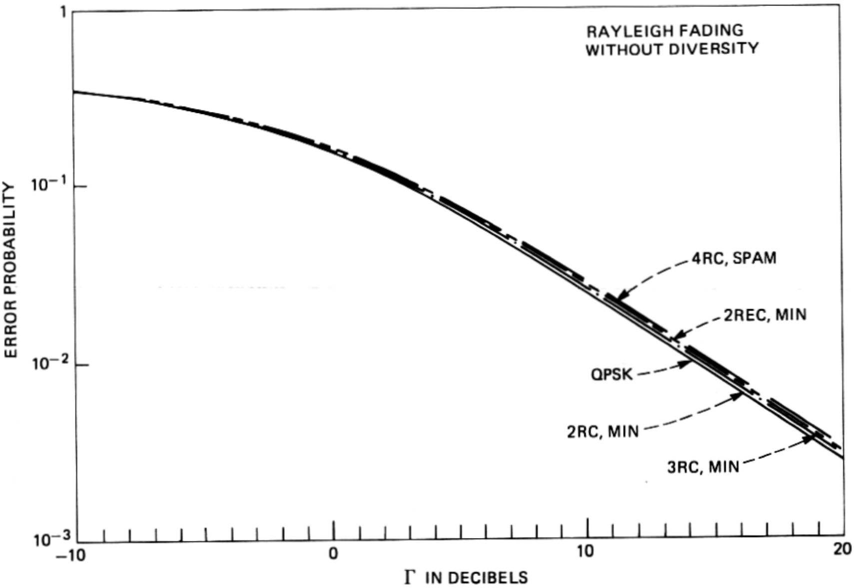


Fig. 11—Error probability P vs s/n Γ for various selected $h = 1/2$, binary CPM schemes with selected receiver filters. On each curve the modulation format and the receiver filter are given.

where P is the error probability in the symbol decision after the linear filter in each quadrature arm. Thus, for the fading and diversity case, the same relative curves are obtained as upper bounds

$$P_{bit} \leq 2P \tag{43}$$

on the bit-error probability with differential encoding/decoding. In absolute numbers, all error probabilities P must be multiplied by 2. In principle, averaging of the type in (17) can be performed numerically for (42) but, especially for large s/n's and large numbers of diversity branches, the bound based on eq. (43) is very tight.

The calculations of the bit-error probability for TFM with fading are given in Ref. 28. This lies, as expected, between that of $3RC$ and $4RC$.

The performance of GMSK in fading for various values of the parameter B_bT—affecting the length of the pulse $g(t)$—can be estimated by comparisons to the raised-cosine pulses.[3,4] Compare the spectra and eye patterns in Refs. 3 and 4 with those for raised-cosine pulses in Refs. 11 through 13. Approximate fading and diversity ($M = 2$) error-probability behavior is given in Ref. 3 for some cases of GMSK. A detailed comparison of raised-cosine schemes and GMSK is given in Ref. 28.

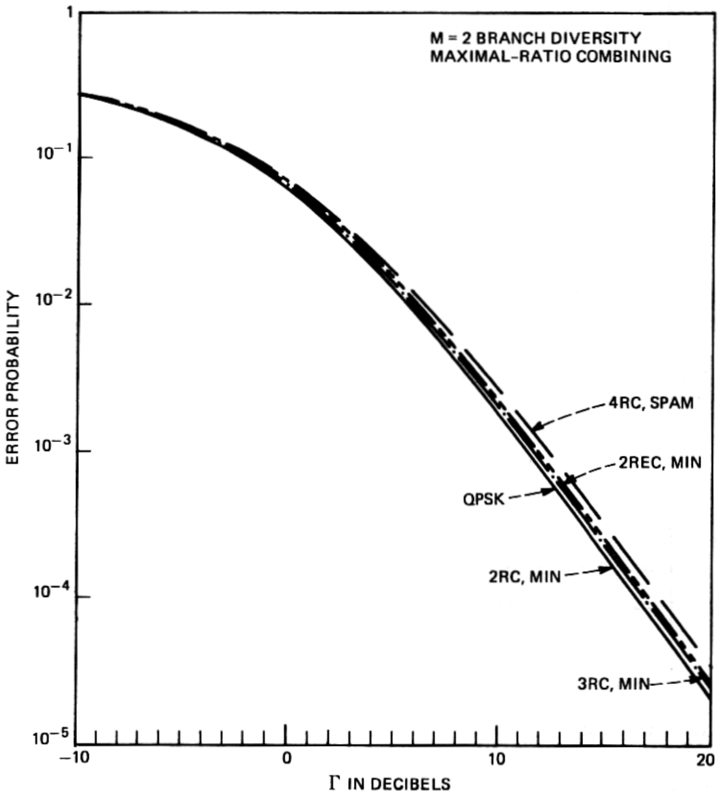


Fig. 12—Error probability P vs average-per-branch s/n Γ for the group of modulation schemes in Fig. 11. $M = 2$ branch diversity with maximal-ratio combining is used.

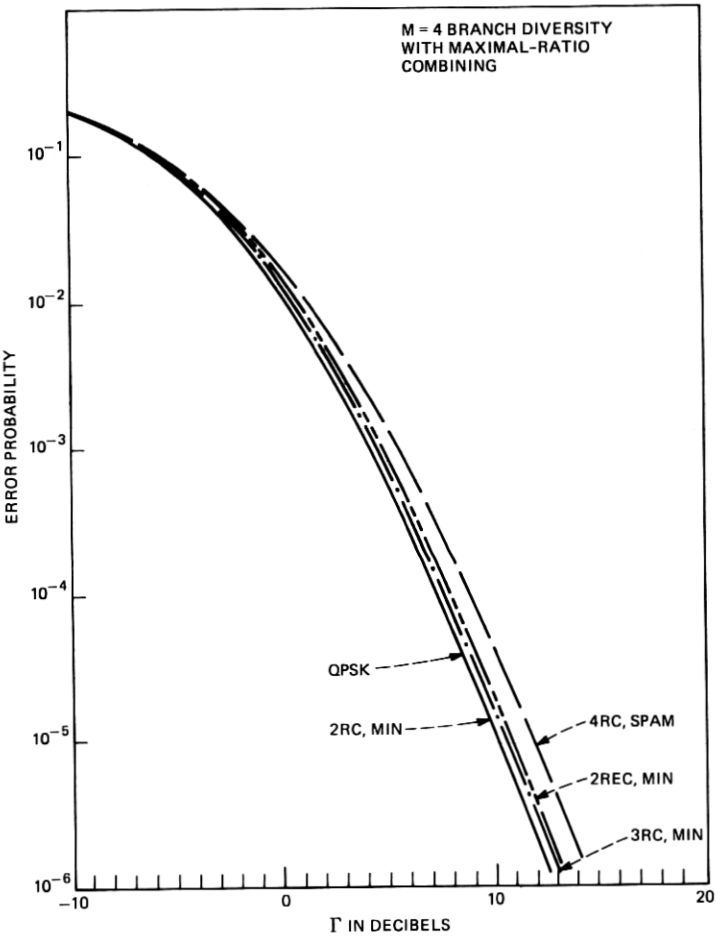


Fig. 13—Error probability P vs average-per-branch s/n Γ for $M = 4$ branch diversity with maximal-ratio combining.

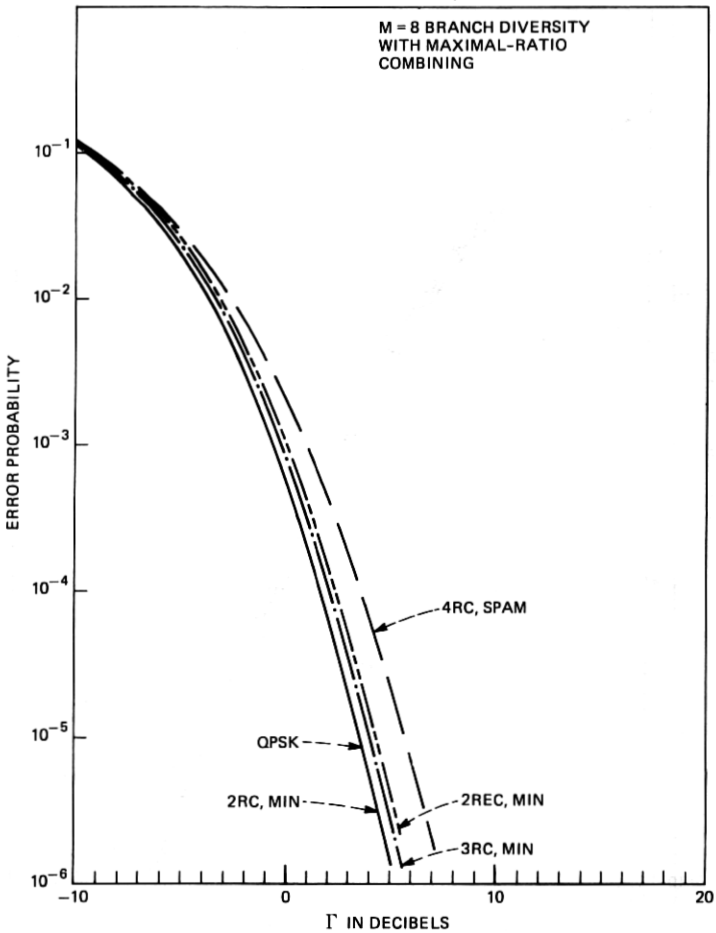


Fig. 14—Error probability P vs average-per-branch s/n Γ for $M = 8$ branch diversity with maximal-ratio combining.

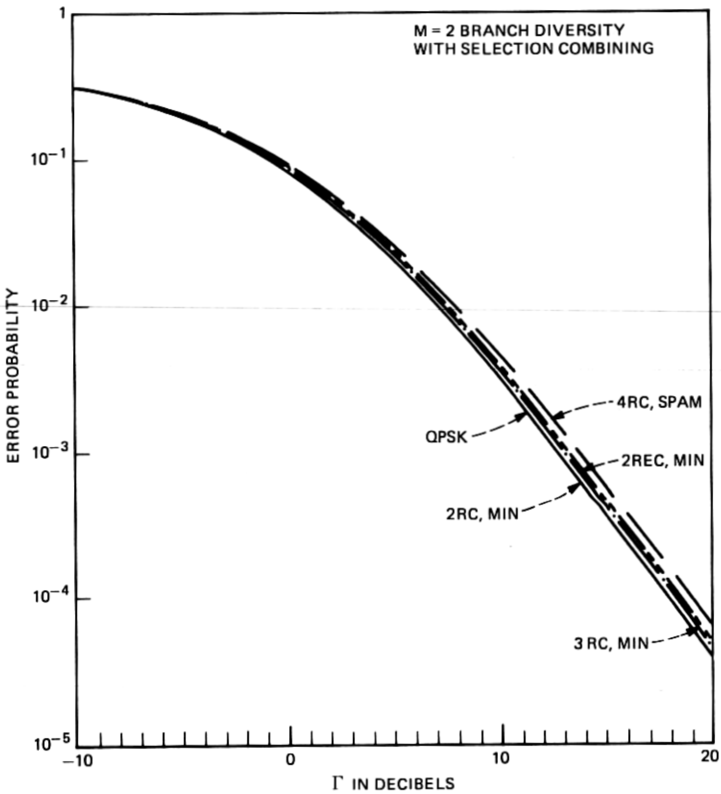


Fig. 15—Error probability P vs average-per-branch s/n Γ for $M = 2$ branch diversity with selection combining.

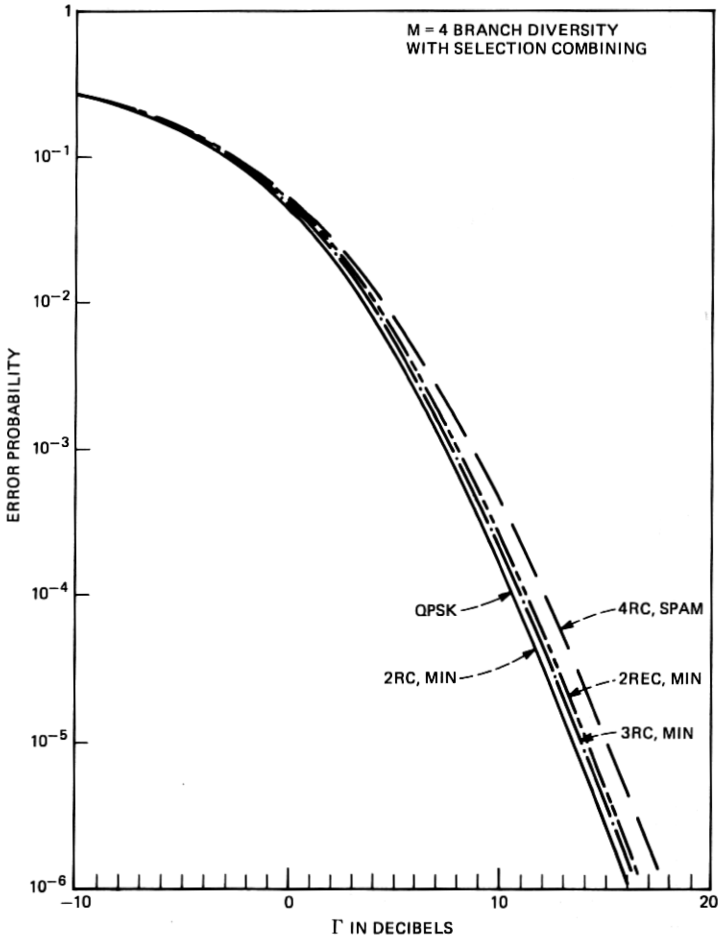


Fig. 16—Error probability P vs average-per-branch s/n Γ for $M = 4$ branch diversity with selection combining.

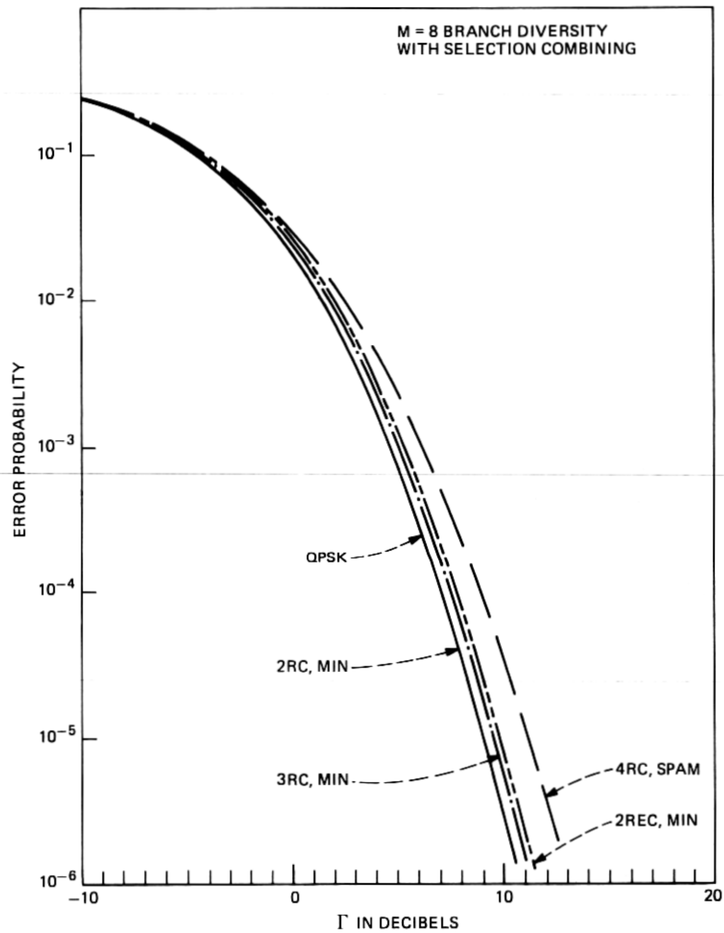


Fig. 17—Error probability P vs average-per-branch s/n Γ for $M = 8$ branch diversity with selection combining.

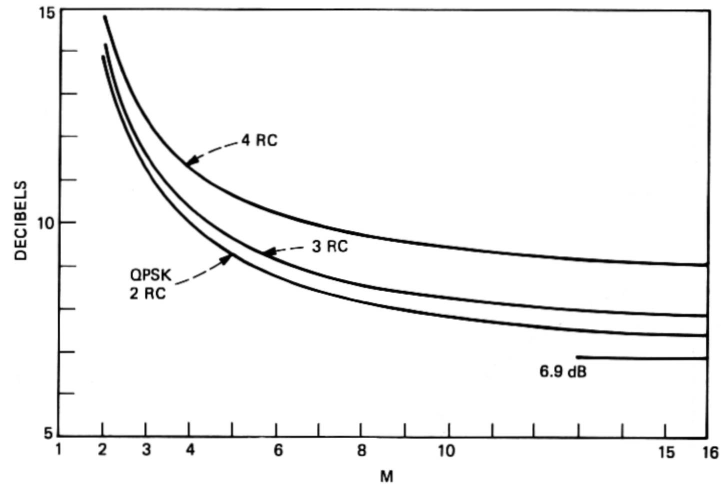


Fig. 18—Required receiver output s/n to achieve 10^{-3} error probability as a function of the number of branches for coherent detection of QPSK and for 2*RC*, 3*RC*, and 4*RC* with SPAM filters.

This paper contains formulas for the calculations of the bit-error probability for a class of modulation schemes with attractive spectral behavior. The class contains such modulation schemes as MSK, FFSK, TFM, and GMSK, as defined above. A simple suboptimum coherent detector can be used with performance approaching the optimum detector. The channel is assumed to be a slow Rayleigh fading channel with Gaussian noise, and diversity is employed to combat multipath fading. We assume the receiver uses either ideal maximal-ratio combining or selection combining. Analytical formulas are derived for both cases, and simple asymptotic expressions for large signal-to-noise ratios are also derived and discussed. It is noted from the numerical calculations and also from the asymptotic formulas that the difference in E_b/N_o between various modulation schemes in the considered class at a given error probability decreases as the number of branches of diversity decreases.

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