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Comparisons on Blocking Probabilities for Regular Series Parallel Channel Graphs

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We give a sufficient condition for one regular series parallel channel graph to be superior to another with the same number of stages. The main mathematical tools used for doing this are the recently developed results on majorization over a partial order.

I. INTRODUCTION

An s-stage channel graph is a graph whose vertices can be partitioned into s subsets (stages) $V_1, V_2, \cdots, V_s$, with V_1 and V_s each containing a single vertex (called the source and the sink, respectively), and whose edges can be partitioned into $s - 1$ subsets $E_1, E_2, \cdots, E_{s-1}$ such that

(*i*) Edges in E_i connect vertices in V_i to vertices in V_{i+1},

(*ii*) Each vertex in V_i, $1 < i < s$, is connected to at least one vertex in each of V_{i-1} and V_{i+1}.

A channel graph is regular if for each i, the numbers of edges in E_{i-1} and E_i coincident to a vertex in V_i are independent of which vertex is chosen.

A series combination of an s-stage channel graph G and a t-stage channel graph H is a union of G and H into an $(s + t - 1)$-stage channel graph, with the sink of G identified with the source of H. A parallel combination of two s-stage channel graphs is a union of these two graphs into another s-stage channel graph with the source and the sink of one graph being identified with the source and the sink, respectively, of the other graph. A channel graph is series parallel if it is either an edge or is constructable from two smaller series parallel channel graphs by either a series or a parallel combination. A series parallel canopy is a special case of a series parallel channel graph in which parallel combinations are allowed only when at least one of the two component subgraphs consists solely of a single edge.

Each edge in a channel graph can be in one of two states, occupied or idle. In this paper, we follow Lee's assumption[1] that the states of the edges are independent and that each edge in E_i has probability p_i, called the occupancy for E_i, of being occupied. The blocking probability of a channel graph is the probability that every *channel*—by which we mean a path from source to sink consisting of one edge from each E_i—contains at least one occupied edge. An s-stage channel graph is said to be superior to another s-stage channel graph if the blocking probability of the former never exceeds that of the latter, independent of the occupancies for the E_i (common to both graphs).

Chung and Hwang[2] showed that a regular series parallel channel graph (hereafter referred to as rspcg) without multiple edges can be uniquely represented by its degree vector. They also proved that in the case of two s-stage regular series parallel canopies, a necessary and sufficient condition for one graph to be superior to the other is that the degree vector of the former "majorizes" that of the latter. They conjectured that the same condition might also hold for rspcg's. However, counterexamples to the sufficiency of the condition for rspcg's were given in Refs. 3 and 4. In this paper, we give a sufficient condition for one s-stage rspcg to be superior to another, with multiple edges between two vertices allowed, by using the recently developed results of majorization over a partial order.[5,6]

II. MAJORIZATION OVER A PARTIAL ORDER

A set of numbers $A = \{a_1 \geq a_2 \geq \cdots \geq a_n\}$ is said to be *weakly submajorized*[7] by another set of numbers $B = \{b_1 \geq b_2 \geq \cdots \geq b_n\}$ if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \qquad \text{for each} \quad k = 1, \cdots, n.$$

If, in addition,

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i,$$

then A is simply said to be majorized by B.

The above concept of set majorization has been extended to majorization over a partial order.[5,6] Let $P = \{S, \rightarrow\}$ denote a partial order on S, where S is a set of n elements and $s_i, s_j \in S$, $s_i \rightarrow s_j$ indicates that s_i is greater than s_j in P. Let $A = \{a_1, a_2, \cdots, a_n\}$ and $B = \{b_1, b_2, \cdots, b_n\}$, where A and B can be thought of as two sets of weights for the elements in S. Then A is said to be weakly submajorized by B on P if for every filter S_j of S,

$$\sum_{s_i \in S_j} a_i \leq \sum_{s_i \in S_j} b_i,$$

where S_j is a filter if $s_i \in S_j$ and $s_k \to s_i \Rightarrow s_k \in S_j$. If equality holds for $S_j = S$, then A is simply said to be majorized by B on P.

Lemma 1. Suppose A is weakly submajorized by B on P. Then there exists $C = \{c_1, c_2, \cdots, c_n\}$, where $c_i \geq 0$ for all i, such that $A + C$ is majorized by B on P.

Proof: The proof is by induction on n. For $n = 1$, Lemma 1 is true by setting $c_1 = b_1 - a_1$. For general n, without loss of generality, assume that s_n is a minimal element in P. Set $c_n = b_n - a_n$. Then $A' = \{a_1, a_2, \cdots, a_{n-1}, a_n + c_n\}$ is still weakly submajorized by B on P, since for any filter S_j containing s_n,

$$\sum_{s_i \in S_j} b_i - \sum_{s_i \in S_j} a'_i = \sum_{s_i \in S_j - \{s_n\}} b_i - \sum_{s_i \in S_j - \{s_n\}} a_i \geq 0.$$

Next consider the partial order P on $S - \{s_n\}$. By our inductive assumption, there exists nonnegative $c_1, c_2, \cdots, c_{n-1}$ such that $(a_1 + c_1, a_2 + c_2, \cdots, a_{n-1} + c_{n-1})$ is majorized by $(b_1, b_2, \cdots, b_{n-1})$. Lemma 1 follows immediately.

We quote a result from Ref. 5:

Theorem 1: Let $f(x_1, x_2, \cdots, x_n)$ be a function defined over the domain D. Let $P = (X, \to)$ denote a partial order, where $X = \{x_1, x_2, \cdots, x_n\}$. Then

$$f(a_1, a_2, \cdots, a_n) \leq f(b_1, b_2, \cdots, b_n)$$

for all A majorized by B on P if and only if f is such that for every i and j,

$$x_i \to x_j \Rightarrow \frac{\partial f}{\partial x_i} \geq \frac{\partial f}{\partial x_j} \qquad \text{over all} \quad X \in D.$$

We now generalize Theorem 1 into Theorem 2.

Theorem 2: Let $f(x_1, x_2, \cdots, x_n)$ be a function defined over the domain D such that f is monotone nonincreasing in each of its arguments. Let $P = (X, \to)$ denote a partial order, where $X = \{x_1, x_2, \cdots, x_n\}$. Then

$$f(a_1, a_2, \cdots, a_n) \geq f(b_1, b_2, \cdots, b_n)$$

for all A weakly submajorized by B on P if and only if f is such that for every i and j,

$$x_i \to x_j \Rightarrow \frac{\partial f}{\partial x_i} \leq \frac{\partial f}{\partial x_j} \qquad \text{over all} \quad X \in D.$$

Proof:

(*i*) Assume $f(a_1, a_2, \cdots, a_n) \geq f(b_1, b_2, \cdots, b_n)$ for all A weakly submajorized by B on P. Then, in particular,

$$f(a_1, a_2, \cdots, a_n) \geq f(b_1, b_2, \cdots, b_n),$$

or equivalently,

$$-f(a_1, a_2, \cdots, a_n) \leq -f(b_1, b_2, \cdots, b_n)$$

for all A majorized by B. From Theorem 1, a necessary condition for this to happen is that for every i and j,

$$x_i \rightarrow x_j \Rightarrow \frac{\partial(-f)}{\partial x_i} \geq \frac{\partial(-f)}{\partial x_j},$$

or equivalently

$$\frac{\partial f}{\partial x_i} \leq \frac{\partial f}{\partial x_j} \qquad \text{over all} \quad X \in D.$$

(*ii*) Assume that for every i and j

$$x_i \rightarrow x_j \Rightarrow \frac{\partial f}{\partial x_i} \leq \frac{\partial f}{\partial x_j} \qquad \text{over all} \quad X \in D.$$

Let A be weakly submajorized by B on P and let $\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i = c \geq 0$. From Lemma 1, there exists nonnegative C such that $A + C$ is majorized by B. From Theorem 1,

$$-f(a_1 + c_1, a_2 + c_2, \cdots, a_n + c_n) \leq -f(b_1, b_2, \cdots, b_n).$$

Since f is monotone nonincreasing in each x_i, it follows that $f(a_1, a_2, \cdots, a_n) \geq f(a_1 + c_1, a_2 + c_2, \cdots, a_n + c_n) \geq f(b_1, b_2, \cdots, b_n)$. ∎

III. THE MAIN RESULTS

Theorem 2 will be used for comparing two rspcg's. To do this, however, we first have to define a partial order such that an rspcg can be represented as a set of weights for the elements of the partial order. This can be done by using the Takagi graph characterization of an rspcg.

An (i, j, r) multiplex, $1 \leq i \leq j \leq s$, of an s-stage channel graph G is an s-stage channel graph formed from the union of r copies of G, with the copies being merged into a single copy from stage 1 to stage i and from stage j to stage s. A channel graph is called a Takagi graph[8,9] if it can be obtained as a multiplex of a smaller Takagi graph, where the smallest Takagi graph of s stages is an s-stage path. An (i, j, r) multiplex can also be represented by the equation $m_{ij} = r$, where m_{ij} is called the *multiplex index* and r is the value of the index. Therefore, a Takagi graph can be represented by a set $\{m_{ij} = k\}$ called a multiplex set. Figure 1 illustrates how the Takagi graph $\{m_{13} = 3, m_{24} = 2\}$ is constructed. It is clear that adding or deleting a multiplex index with value one has no effect on the Takagi graph. Up to this

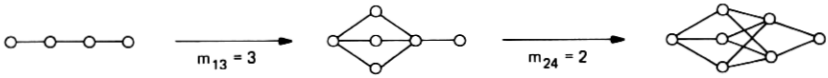

Fig. 1—A Takagi graph.

equivalence, it has been proved (see Ref. 4) that there exists a one-to-one mapping between multiplex sets and Takagi graphs, regardless of the ordering of the multiplex indices in the set. Furthermore, it is straightforward to verify that for an rspcg the product of all the values in its multiplex set equals the total number of distinct channels.

Let m_{ij} and m_{pq} denote two multiplex indices. Then m_{ij} is said to *cross* m_{pq} if $i < p < j < q$, and to *contain* m_{pq} if $i \le p < q \le j$. The following has been proved in Ref. 10:

Theorem 3: A channel graph is an rspcg if and only if it is a Takagi graph without crossing multiplex indices.

We define a partial order P_{1s} on the set of multiplex indices $\{m_{ij}: 1 \le i < j \le s\}$ by: $m_{ij} \to m_{pq}$ if m_{ij} contains m_{pq}. Then the multiplex set of any s-stage rspcg can be considered as a set of weights for the elements of P_{1s} (if m_{ij} is not in the multiplex set, we define $m_{ij} = 1$). For a given multiplex set M, we define M_{ij} to be the subset of M consisting of all multiplex indices contained by m_{ij}. We also let P_{ij} denote the partial order P restricted on M_{ij}. For fixed occupancies p_1, $p_2, \cdots, p_{s-1}$, let $B(M)$ denote the blocking probability for the Takagi graph with multiplex set M. Then, from Theorem 3, we have

$$B(M_{ij}) = \left\{1 - \prod_{l \in L_{ij}} (1 - p_l) \prod_{m_{pq} \in N_{ij}} [1 - B(M_{pq})]\right\}^{m_{ij}},$$

where $L_{ij} = \{l: m_{l,l+1} = 1,\ m_{ij} \to m_{l,l+1}$, but there does not exist $m_{uv} > 1$ such that $m_{ij} \to m_{uv} \to m_{l,l+1}\}$ and where $N_{ij} = \{m_{pq}: m_{pq} > 1,$ $m_{ij} \to m_{pq}$, but there does not exist $m_{uv} > 1$ such that $m_{ij} \to m_{uv} \to m_{pq}\}$. We quote a result from Ref. 2:

Lemma 2: For given constants $c_1, c_2, \cdots, c_n$, all lying between zero and one, define

$$f(x_n) = (1 - c_n)^{x_n};$$

$$f(x_k, x_{k+1}, \cdots, x_n) = \{1 - c_k[1 - f(x_{k+1}, x_{k+2}, \cdots, x_n)]\}^{x_k}$$

$$\text{for} \quad k = 1, 2, \cdots, n-1.$$

Suppose the vector $(\ln a_1, \ln a_2, \cdots, \ln a_n)$ is weakly submajorized by the vector $(\ln b_1, \ln b_2, \cdots, \ln b_n)$ where $\ln a_i$ and $\ln b_i$ are nonnegative for all i. Then

$$f(a_1, a_2, \cdots, a_n) \ge f(b_1, b_2, \cdots, b_n).$$ ∎

In particular, for any $w > 1$ and $i < j$, we have $f(a_1, a_2, \cdots, a_i, \cdots, a_j w, \cdots, a_n) \geq f(a_1, a_2, \cdots, a_i w, \cdots, a_j, \cdots, a_n)$. Therefore, we also have:

Corollary:

$$\frac{\partial f(x_1, x_2, \cdots, x_n)}{\partial \ln x_i} \leq \frac{\partial f(x_1, x_2, \cdots, x_n)}{\partial \ln x_j} \quad \textit{for} \quad i \leq j.$$

We are now ready to prove Theorem 4.

Theorem 4: An s-stage rspcg with the multiplex set $\{m_{ij} = a_{ij}\}$ is superior to another s-stage rspcg with the multiplex set $\{m_{ij} = b_{ij}\}$ if $\{\ln b_{ij}\}$ is weakly submajorized by $\{\ln a_{ij}\}$ on P_{1s}.

Proof: A straightforward induction proof shows that $B(M_{1s})$ is monotone nonincreasing in each $m_{ij} \in M_{1s}$. Therefore, if we can prove that for every $m_{uv} \rightarrow m_{xy}$,

$$\frac{\partial B(M_{1s})}{\partial \ln m_{uv}} \leq \frac{\partial B(M_{1s})}{\partial \ln m_{xy}},$$

then Theorem 4 will follow immediately from Theorem 2.

Consider a path Z from the top of P_{1s} to the bottom of P_{1s} which contains m_{uv} and m_{xy}. Let r_i, $i = 1, 2, \cdots, n$, denote the value of the ith multiplex index on this path. Suppose we hold every other m_{ij} constant except those on Z. Then $B(M_{1s})$ can be expressed as a function of $r_1, r_2, \cdots, r_n$ alone since all other m_{ij} are now constants. To be more specific, we have

$$B(r_n) = (1 - c_n)^{r_n}$$

and

$$B(r_k, r_{k+1}, \cdots, r_n) = \{1 - c_k[1 - B(r_{k+1}, r_{k+2}, \cdots, r_n)]\}^{r_k}$$
$$\text{for} \quad k = 1, 2, \cdots, n-1.$$

From the Corollary of Lemma 2, we conclude that $i < j$ implies

$$\frac{\partial B(r_1, r_2, \cdots, r_n)}{\partial \ln r_i} \leq \frac{\partial B(r_1, r_2, \cdots, r_n)}{\partial \ln r_j}.$$

In particular, we have

$$\frac{\partial B(r_1, r_2, \cdots, r_n)}{\partial \ln m_{uv}} \leq \frac{\partial B(r_1, r_2, \cdots, r_n)}{\partial \ln m_{xy}}.$$

The proof is now complete by noting

$$\frac{\partial B(M_{1s})}{\partial \ln m_{uv}} = \frac{\partial B(r_1, r_2, \cdots, r_n)}{\partial \ln m_{uv}}$$

and

$$\frac{\partial B(M_{1s})}{\partial \ln m_{xy}} = \frac{\partial B(r_1, r_2, \cdots, r_n)}{\partial \ln m_{xy}}.$$

Define $\bar{M}_{1s} = \{m_{ij} \in M_{1s}\colon L_{ij} \neq \phi\}$ and define the partial order $\bar{P}_{1s}$ accordingly.

Theorem 5: An s-stage rspcg with the multiplex set $\{m_{ij} = a_{ij}\}$ is superior to another s-stage rspcg with the multiplex set $\{m_{ij} = b_{ij}\}$ only if $\{\ln b_{ij}\}$ is weakly submajorized by $\{\ln a_{ij}\}$ on $\bar{P}_{1s}$ (associated with the $\{a_{ij}\}$ set).

Proof: Consider two s-stage rspcg's A and B with multiplex numbers $\{m_{ij} = a_{ij}\}$ and $\{m_{ij} = b_{ij}\}$, respectively. Suppose there exists a filter $M \subset \bar{M}_{1s}$ such that

$$\sum_{m_{ij} \in M} \ln a_{ij} < \sum_{m_{ij} \in M} \ln b_{ij}.$$

Consider a set of occupancies $p_1, p_2, \cdots, p_{s-1}$ such that $p_k = 0$ if $m_{k,k+1}$ is contained by any m_{ij} not in M. Clearly, if all edges from stage i to stage j are idle, then we can set a_{ij} and b_{ij} to 1 without affecting the blocking probabilities of A and B. But now, owing to the assumption

$$\sum_{m_{ij} \in M} \ln a_{ij} < \sum_{m_{ij} \in M} \ln b_{ij},$$

the product of all a_{ij} in A is less than the product of all b_{ij} in B, or equivalently, there are fewer paths in A than in B. But it is well known that when the occupancies of all edges approach one,[4] then the blocking

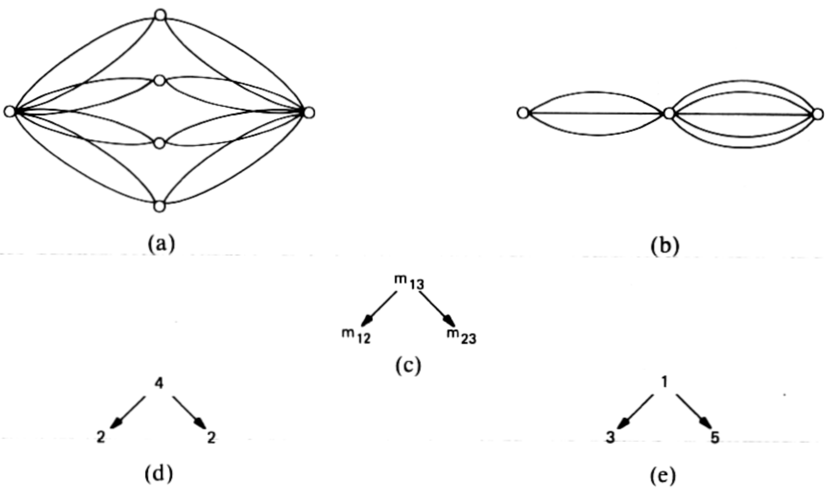

Fig. 2—Graphs for Example 1.

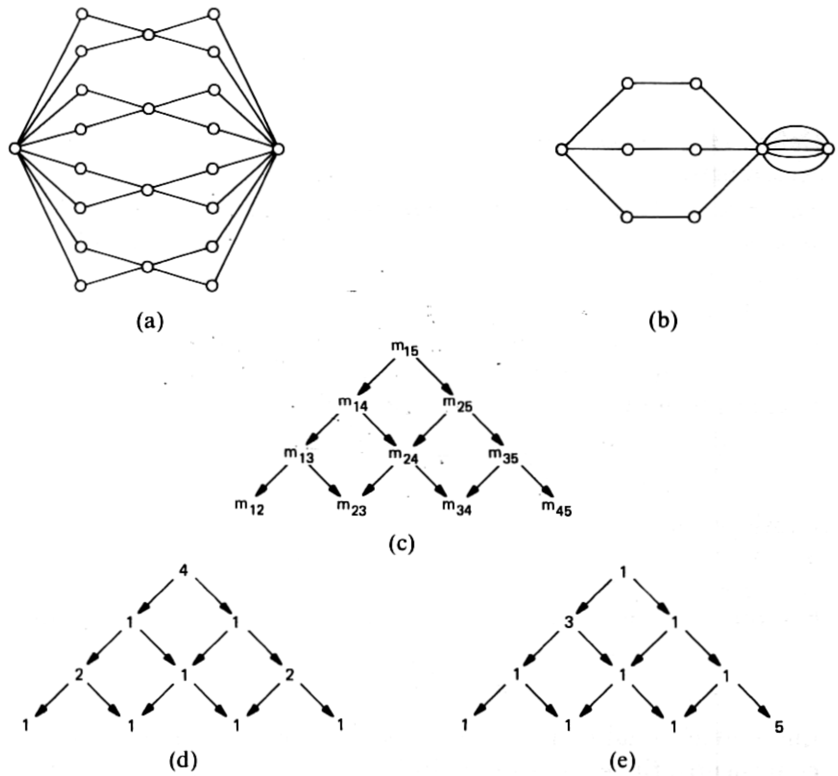

Fig. 3—Graphs for Example 2.

probability of a channel graph with fewer channels exceeds the blocking probability of a channel graph with more channels. Therefore, A cannot be superior to B. ∎

IV. EXAMPLES

Example 1. Figure 2(a) shows the Takagi graph $A = \{m_{12} = m_{23} = 2, m_{13} = 4\}$. Figure 2(b) shows the Takagi graph $B = \{m_{12} = 3, m_{23} = 5\}$. Figure 2(c) shows the partial order P_{13}. Figure 2(d) shows the weights of A on P_{13}. Figure 2(e) shows the weights of B on P_{13}.

It is easily seen that M_{13} has only four filters, $\{m_{13}\}$, $\{m_{12}, m_{13}\}$, $\{m_{23}, m_{13}\}$ and $\{m_{12}, m_{23}, m_{13}\}$, and product of the weights of A is greater than that of B in every case. From Theorem 4, the first graph is superior to the second graph.

Example 2. Figure 3(a) shows the Takagi graph $A = \{m_{15} = 4, m_{13} = m_{35} = 2\}$. Figure 3(b) shows the Takagi graph $B = \{m_{14} = 3, m_{45} = 5\}$. Figure 3(c) shows the partial order P_{15}. Figure 3(d) shows the weights of A on P_{15}. Figure 3(e) shows the weights of B on P_{15}.

Consider the filter $M = \{m_{14}, m_{45}, m_{35}, m_{15}, m_{25}\}$. The product of the weights of A on M is 8 while the product of the weights of B on M is 15. Hence, A is not superior to B. Note that A can still be preferable to B (or B preferable to A) in many other senses. But one does not dominate the other as far as the strong property of superiority is concerned.

V. CONCLUSION

Channel graphs, of which regular series parallel channel graphs form an important subclass, have been extensively used in modeling and analyzing blocking probabilities of switching networks. A popular concept in comparing the blocking characteristics of two channel graphs is to see whether one is superior to the other under arbitrary traffic loads. We give a sufficient condition for superiority in comparing regular series parallel channel graphs.

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