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Approximate Mean Waiting Times in Transient *GI/G/1* Queues

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In this paper we give an approximation method for obtaining the probability the server is busy and the mean waiting time as seen by the nth arriving customer for the GI/G/1 queueing system. Transient behavior is the key issue of the method. The approximation consists of a pair of recursion formulae whose state variables are the probability of delay and the mean waiting time. Any initial state may be prescribed for the 0th arriving customer. Programming is very easy, and the computation is rapid. The procedure is useful for rush-hour analyses and for studying the recovery of a system from temporary overload.

I. INTRODUCTION

This paper presents an approximation method for obtaining the probability the server is busy and the mean waiting time as seen by the nth arriving customer for the $GI/G/1$ queueing system. Thus, transient behavior is the key issue of the method. The approximation consists of a simultaneous pair of recursion formulae whose state variables are the probability of delay and the mean waiting time. It is assumed that the 0th arriving customer finds the queue in some prescribed state from which the successive states are computed. Naturally, for the nth arrival when n is large, the computations provide approximations to the corresponding eqilibrium quantities when equilibrium exists. The procedure, however, is not limited to queues possessing an equilibrium state. For methods specially adapted to approximating the equilibrium quantities, we refer to a paper by A. A. Fredericks,[1] in which the approximation $A_{0,1}$ of that paper essentially corresponds to the equilibrium results obtained here; we also refer to Fredericks for an application to computer systems.[2]

Transient analysis is of particular importance in studying the recovery of a system from temporary overload. This can occur after a short downtime during which the buffers become full and the machine must now work off the accumulated load. Another situation calling for transient analysis occurs in the case of minicomputers receiving hourly data from an electronic switching system (ESS). The hourly rush of traffic requires a transient analysis to determine the buildup and falloff of delays. This corresponds to the general problem of rush-hour situations. Knowledge of the transient behavior of a queue also indicates the number of customers needed in the arrival stream until approximate equilibrium conditions are recovered from some temporary overload (i.e., the relaxation time may be estimated).

In Section II we obtain the general recursion formulae and the corresponding formulae defining the equilibrium results. In Section III we specialize the recursions to the $GI/M/1$ queue and present the exact solutions obtained using the Takaćs[3] method as a control. We show the relationship between the exact probability of delay and mean waiting time, and also conclude that the approximate equilibrium quantities are, in fact, exact. We discuss a number of numerical examples and compare them with exact results. In Section IV we reduce the general recursions to the $M/G/1$ case. A relation is again obtained between the probability of delay and the mean waiting time. The interesting property is proved that the approximate probability of delay and mean waiting time as seen by the nth arrival satisfies the same relation as the exact quantities; thus, the approximation method is shown to preserve certain properties of the exact solution. We discuss a number of numerical examples and compare them with exact results. In Section V we present several numerical examples of $GI/G/1$ queues. Exact solutions, however, are derived only for equilibrium values.

In all the numerical examples given, the queues start empty; however, this is done only to facilitate obtaining the controls. Any initial state may, of course, be used in the recursion formulae. A synopsis of the important recursions and definitions of symbols is given in Section VI.

II. RECURSION FORMULAE

Let $A(x)$ be the distribution of time between arrivals with mean arrival rate λ, and $B(x)$ the distribution of service time with mean service rate μ; the random variable ξ, which is service time minus interarrival time, has the distribution

$$K(x) = \int_{0_-}^{\infty} B(x+y)dA(y). \tag{1}$$

If W_n is the waiting time of the nth arrival, then we have the recursion

$$W_n = [W_{n-1} + \xi_n]^+, \tag{2}$$

in which

$$\begin{aligned} [x]^+ &= x, \qquad x \geq 0 \\ &= 0, \qquad x \leq 0 \end{aligned}$$

and the ξ_n are iid with the common distribution $K(x)$. Let $W_n(x)$ be the distribution function of W_n; then $\hat{W}_n(s)$ is defined by

$$\hat{W}_n(s) = \int_{0_-}^{\infty} e^{-sx} dW_n(x), \tag{3}$$

and hence

$$\hat{W}_n(s) = Ee^{-sW_n}. \tag{4}$$

A recursion relating $\hat{W}_n(s)$ to $\hat{W}_{n-1}(s)$ will now be developed. From eqs. (2) and (4) we have

$$\hat{W}_n(s) = Ee^{-s(W_{n-1}+\xi_n)^+}, \tag{5}$$

and hence

$$\hat{W}_n(s) = E\int_{-\infty}^{\infty} e^{-s(W_{n-1}+x)^+} dK(x), \tag{6}$$

in which the expectation is over the distribution of W_{n-1}. Let

$$\hat{K}(s) = \int_{-\infty}^{\infty} e^{-sx} dK(x); \tag{7}$$

then the functions

$$\hat{K}_+(s) = \int_{0_-}^{\infty} e^{-sx} dK(x) \tag{8}$$

and

$$\hat{K}_-(s) = \int_{-\infty}^{0_-} e^{-sx} dK(x) \tag{9}$$

will be used in the development. From eq. (6), we find that

$$\hat{W}_n(s) = \hat{W}_{n-1}(s)\hat{K}_+(s) + \int_{-\infty}^{0_-} Ee^{-s[W_{n-1}+x]^+} dK(x). \tag{10}$$

The recursion of eq. (10) is exact but, to obtain a simple, explicit recursion, an approximation will be used at this point for the distribution of W_{n-1}. The more accurate the approximation chosen, the more accurate the final recursion will be, but also, presumably, the more difficult to use; accordingly, a simple exponential approximation will be used.

Let

$$\alpha_n = EW_n, \tag{11}$$

and

$$J_n = 1 - W_n(0+); \tag{12}$$

then we use the approximation

$$\begin{aligned} W_n(x) &\simeq 1 - J_n e^{-\frac{J_n}{\alpha_n}x}, \qquad x \geq 0, \\ &= 0, \quad x < 0. \end{aligned} \tag{13}$$

From eq. (13) we compute

$$Ee^{-s[W_{n-1}+x]^+} = 1 - \frac{J_{n-1}s}{\frac{J_{n-1}}{\alpha_{n-1}} + s} e^{\frac{J_{n-1}}{\alpha_{n-1}}x}, \qquad x < 0. \tag{14}$$

Thus, eqs. (10) and (14) yield the recursion

$$\hat{W}_n(s) = \hat{W}_{n-1}(s)\hat{K}_+(s) + \hat{K}_-(0) - \frac{J_{n-1}s}{\frac{J_{n-1}}{\alpha_{n-1}} + s}\hat{K}_-\left(-\frac{J_{n-1}}{\alpha_{n-1}}\right). \tag{15}$$

A pair of recursion relations will now be obtained for J_n, α_n. Since

$$\lim_{s\to\infty} \hat{W}_n(s) = 1 - J_n, \tag{16}$$

$$-\hat{W}'_n(0) = \alpha_n, \tag{17}$$

and

$$\lim_{s\to\infty} \hat{K}_+(s) = 0, \tag{18}$$

we obtain from eq. (15)

$$J_n = \hat{K}_+(0) + J_{n-1}\hat{K}_-\left(-\frac{J_{n-1}}{\alpha_{n-1}}\right), \tag{19}$$

and

$$\alpha_n = \left[\hat{K}_+(0) + \hat{K}_-\left(-\frac{J_{n-1}}{\alpha_{n-1}}\right)\right]\alpha_{n-1} - \hat{K}'_+(0). \tag{20}$$

Equations (19) and (20) constitute the approximate recursions sought.

If the queueing system possesses equilibrium values, then approximations designated by J, α, $\hat{W}(s)$ are obtained from

$$J = \hat{K}_+(0) + J\hat{K}_-\left(-\frac{J}{\alpha}\right), \tag{21}$$

$$\alpha = \left[\hat{K}_+(0) + \hat{K}_-\left(-\frac{J}{\alpha}\right)\right]\alpha - \hat{K}'_+(0), \tag{22}$$

and

$$\hat{W}(s) = \frac{1}{1 - \hat{K}_+(s)}\left[\hat{K}_-(0) - \frac{Js}{\frac{J}{\alpha} + s}\hat{K}_-\left(-\frac{J}{\alpha}\right)\right]. \tag{23}$$

The recursions of eqs. (15), (19), and (20), and all subsequent recursions derived from them, apply to arbitrary initial conditions and to stable or unstable queues.

III. *GI/M/1* QUEUEING SYSTEM

To ascertain the quality of the approximations obtained by eqs. (19) and (20), a control is needed. This will be provided by the explicit solution formulated by Takaćs[3] for the $GI/G/1$ system. Accordingly, let

$$G(z, s) = \sum_{n=0}^{\infty} \hat{W}_n(s)z^n, \tag{24}$$

$$\hat{A}(s) = \int_0^{\infty} e^{-sx}dA(x), \tag{25}$$

and

$$\hat{B}(s) = \int_0^{\infty} e^{-sx}dB(x); \tag{26}$$

then

$$\hat{K}(s) = \hat{A}(-s)\hat{B}(s), \tag{27}$$

and the factorization

$$1 - z\hat{K}(s) = \Gamma_+(z, s)\Gamma_-(z, s) \tag{28}$$

defines the functions $\Gamma_+(z, s)$, $\Gamma_-(z, s)$. The function $\Gamma_+(z, s)$ is to be analytic for $R_e s > 0$ and not to vanish for $R_e s \geq 0$, while $\Gamma_-(z, s)$ is to be analytic for $R_e s < 0$ and not to vanish for $R_e s \leq 0$. Further, define the projection operator T by

$$TEe^{-s\eta} = Ee^{-s\eta+}, \tag{29}$$

in which η is an arbitrarily given random variable. Then the exact solution for the generating function, $G(x, s)$, is

$$G(z, s) = \frac{1}{\Gamma_+(z, s)} T\left[\frac{\hat{W}_0(s)}{\Gamma_-(z, s)}\right]. \tag{30}$$

The only exact solutions to be studied in this paper will correspond to the queueing system starting empty; hence, $\hat{W}_0(s) \equiv 1$. For this case the projection of eq. (30) may be evaluated; we find that

$$G(z, s) = \frac{1}{\Gamma_+(z, s)\Gamma_-(z, 0)}. \tag{31}$$

To apply eq. (31) to the $GI/M/1$, we consider

$$\hat{K}(s) = \hat{A}(-s)\frac{\mu}{\mu + s} \tag{32}$$

and

$$1 - z\hat{K}(s) = \frac{\mu + s - \mu z\hat{A}(-s)}{\mu + s}. \tag{33}$$

Let $\delta(z) > 0$ be defined by

$$\mu - \delta - \mu z\hat{A}(\delta) = 0; \tag{34}$$

then we find that

$$1 - z\hat{K}(s) = \frac{\delta + s}{\mu + s}\frac{\mu + s - \mu z\hat{A}(-s)}{\delta + s}. \tag{35}$$

Thus,

$$\Gamma_+(z, s) = \frac{\delta + s}{\mu + s}, \tag{36}$$

$$\Gamma_-(z, s) = \frac{\mu + s - \mu z\hat{A}(-s)}{\delta + s}, \tag{37}$$

and

$$\Gamma_-(z, 0) = \frac{\mu(1 - z)}{\delta}. \tag{38}$$

Hence, from eq. (31), we obtain

$$G(z, s) = \frac{\delta}{\mu(1 - z)}\frac{\mu + s}{\delta + s}. \tag{39}$$

Let j_n, a_n designate the exact values of $1 - W_n(0+)$, and EW_n, respectively, and let

$$j(z) = \sum_{n=0}^{\infty} j_n z^n, \tag{40}$$

$$a(z) = \sum_{n=0}^{\infty} a_n z^n; \tag{41}$$

then, from

$$j(z) = \frac{1}{1-z} - G(z, \infty), \tag{42}$$

$$a(z) = -\frac{\partial}{\partial s} G(z, s)|_0 \tag{43}$$

and eq. (39), we get

$$j(z) = \frac{1 - \delta/\mu}{1-z}, \tag{44}$$

and

$$a(z) = \frac{\delta^{-1} - \mu^{-1}}{1-z}. \tag{45}$$

Let $L_n = j_n - j_{n-1}$, $A_n = a_n - a_{n-1}$ and $L(z), A(z)$ be the corresponding generating functions; then elimination of δ in eqs. (44) and (45) yields

$$A(z) = \frac{1}{\mu}\frac{L(z)}{1 - L(z)} \tag{46}$$

and

$$a(z) = \frac{1}{\mu}\frac{j(z)}{1 - L(z)}. \tag{47}$$

In particular eq. (47) implies the recursion relating j_n, a_n, as follows:

$$a_n = \frac{1}{\mu} j_n + \sum_{k=1}^{n-1} A_{n-k}(j_k - j_{k-1}). \tag{48}$$

From eq. (32), we find that

$$\hat{K}_+(s) = \frac{\mu}{\mu + s}\hat{A}(\mu), \tag{49}$$

$$\hat{K}_-(s) = \frac{\mu}{\mu + s}[\hat{A}(-s) - \hat{A}(\mu)], \tag{50}$$

and

$$\hat{K}_+(0) = \hat{A}(\mu), \qquad \hat{K}'_+(0) = -\frac{1}{\mu}\hat{A}(\mu). \tag{51}$$

Hence, eqs. (19) and (20) specialized to the $GI/M/1$ are

$$J_n = \hat{A}(\mu) + \frac{J_{n-1}}{1 - \frac{1}{\mu}\frac{J_{n-1}}{\alpha_{n-1}}}\left[\hat{A}\left(\frac{J_{n-1}}{\alpha_{n-1}}\right) - \hat{A}(\mu)\right] \tag{52}$$

and

$$\alpha_n = \left\{\hat{A}(\mu) + \frac{1}{1 - \frac{1}{\mu}\frac{J_{n-1}}{\alpha_{n-1}}}\left[\hat{A}\left(\frac{J_{n-1}}{\alpha_{n-1}}\right) - \hat{A}(\mu)\right]\right\}\alpha_{n-1} + \frac{1}{\mu}\hat{A}(\mu). \tag{53}$$

We will now show that the limiting forms of J_n, α_n, that is J, α are exact; thus $J = j$, $\alpha = a$. Equations (52) and (53) show that

$$J = \hat{A}(\mu) + \frac{J}{1 - \frac{1}{\mu}\frac{J}{\alpha}}\left[\hat{A}\left(\frac{J}{\alpha}\right) - \hat{A}(\mu)\right], \tag{54}$$

and

$$\alpha = \left\{\hat{A}(\mu) + \frac{1}{1 - \frac{1}{\mu}\frac{J}{\alpha}}\left[\hat{A}\left(\frac{J}{\alpha}\right) - \hat{A}(\mu)\right]\right\}\alpha + \frac{1}{\mu}\hat{A}(\mu). \tag{55}$$

Hence,

$$\frac{\hat{A}\left(\frac{J}{\alpha}\right) - \hat{A}(\mu)}{1 - \frac{1}{\mu}\frac{J}{\alpha}} = 1 - \frac{1}{J}\hat{A}(\mu) \tag{56}$$

and, from eqs. (55) and (56),

$$\alpha = \frac{1}{\mu}\frac{J}{1 - J} \tag{57}$$

and

$$J = \hat{A}[\mu(1 - J)]. \tag{58}$$

Since eqs. (57) and (58) are the exact relations defining j, a, the statement is proved.

An $M/M/1$ will be used to start the numerical examples. For the $M/M/1$, one has ($\rho = \lambda/\mu$)

$$J_n = \frac{1}{1 + \rho}\left(\rho + \frac{J_{n-1}}{1 + \frac{1}{\lambda}g_{n-1}}\right) \tag{59}$$

and

$$\alpha_n = \frac{1}{1 + \rho}\left[\left(\rho + \frac{1}{1 + \frac{1}{\lambda}g_{n-1}}\right)\alpha_{n-1} + \frac{\rho}{\mu}\right],$$

in which the designation $g_n = J_n/\alpha_n$ will henceforth be used.

Since $J_0 = 0$, $\alpha_0 = 0$, the computations are started with

$$J_1 = \hat{K}_+(0) = \frac{\rho}{1 + \rho}, \tag{60}$$

and

Table I—$M/M/1$, $\rho = 0.2$.

n	J	α	j	a
1	0.1667	0.1667	0.1667	0.1667
2	0.1898	0.2176	0.1898	0.2176
3	0.1962	0.2368	0.1962	0.2364
4	0.1985	0.2445	0.1985	0.2440
5	0.1994	0.2477	0.1993	0.2473

Table II—$M/M/1$, $\rho = 0.8$.

n	J	α	j	a
1	0.4444	0.4444	0.4444	0.4444
2	0.5542	0.7517	0.5542	0.7517
3	0.6047	0.9959	0.6084	0.9912
4	0.6354	1.2016	0.6418	1.1890
5	0.6570	1.3804	0.6650	1.3578

Table III—$M/M/1$, $\rho = 2.0$.

n	J	α	j	a
1	0.6667	0.6667	0.6667	0.6667
2	0.8148	1.2593	0.8148	1.2593
3	0.8719	1.8233	0.8807	1.8189
4	0.9012	2.3727	0.9172	2.3603
5	0.9191	2.9131	0.9362	2.8922

$$\alpha_1 = -K'_+(0) = \frac{1}{\mu}\frac{\rho}{1+\rho}, \tag{61}$$

which are exact. The value $\mu = 1$ will be used in all examples. Tables I through III below present approximate and exact values for $\rho = 0.2$, 0.8, and 2.0.

For the next example, the renewal stream is

$$\hat{A}(s) = \frac{1}{(1+s)(1+2s)}. \tag{62}$$

The recursion relations are

$$J_n = \frac{1}{6}\left[1 + J_{n-1}\frac{5 + 2g_{n-1}}{(1+g_{n-1})(1+2g_{n-1})}\right], \tag{63}$$

and

$$\alpha_n = \frac{1}{6}\left[1 + \frac{5 + 2g_{n-1}}{(1+g_{n-1})(1+2g_{n-1})}\right]\alpha_{n-1} + \frac{1}{6}. \tag{64}$$

Some numerical results are given in Table IV.

As a further example the case $D/M/1$ is considered. We have

$$\hat{A}(s) = e^{-sT}. \tag{65}$$

The recursion relations are

Table IV—$GI/M/1$, $\rho = 1/3$.

n	J	α	j	a
1	0.1667	0.1667	0.1667	0.1667
2	0.1991	0.2269	0.1991	0.2269
3	0.2100	0.2538	0.2101	0.2534
4	0.2147	0.2670	0.2148	0.2662
∞	0.2192	0.2808	0.2192	0.2808

Table V—$D/M/1$, $T = 2$, $\rho = 0.5$.

n	J	α	j	a
1	0.1353	0.1353	0.1353	0.1353
2	0.1720	0.1903	0.1720	0.1903
3	0.1867	0.2179	0.1868	0.2175
4	0.1938	0.2331	0.1940	0.2324
5	0.1977	0.2419	0.1978	0.2410
∞	0.2032	0.2550	0.2032	0.2550

$$J_n = e^{-T} + J_{n-1}\frac{e^{-Tg_{n-1}} - e^{-T}}{1 - g_{n-1}}, \tag{66}$$

and

$$\alpha_n = \left(e^{-T} + \frac{e^{-Tg_{n-1}} - e^{-T}}{1 - g_{n-1}}\right)\alpha_{n-1} + e^{-T}. \tag{67}$$

We find that the exact result for j_n is

$$j_n = \sum_{k=1}^{n} \frac{k^{k-1}}{k!} T^{k-1}e^{-kT}. \tag{68}$$

The equilibrium values for $T = 2$ are $J = j = 0.2032$, $\alpha = a = 0.2550$. Table V presents the numerical results.

V. $M/G/1$ QUEUEING SYSTEM

For the $M/G/1$ queue, we have

$$\hat{K}(s) = \frac{\lambda}{\lambda - s}\hat{B}(s); \tag{69}$$

hence

$$1 - z\hat{K}(s) = \frac{\lambda - s - \lambda z\hat{B}(s)}{\lambda - s}. \tag{70}$$

Define $\delta(z) > 0$ by

$$\lambda - \delta - \lambda z\hat{B}(\delta) = 0; \tag{71}$$

then

$$1 - z\hat{K}(s) = \frac{\delta - s}{\lambda - s}\frac{\lambda - s - \lambda z\hat{B}(s)}{\delta - s}, \tag{72}$$

and hence

$$\Gamma_+(z, s) = \frac{\lambda - s - \lambda z \hat{B}(s)}{\delta - s}, \tag{73}$$

$$\Gamma_-(z, s) = \frac{\delta - s}{\lambda - s}, \tag{74}$$

and

$$\Gamma_-(z, 0) = \frac{\delta}{\lambda}. \tag{75}$$

Accordingly, the generating function, $G(z, s)$, is

$$G(z, s) = \frac{\lambda}{\delta} \frac{\delta - s}{\lambda - s - \lambda z \hat{B}(s)}; \tag{76}$$

thus, the generating functions $j(z)$, $a(z)$ are

$$j(z) = \frac{1}{1 - z} - \frac{\lambda}{\delta}, \tag{77}$$

and

$$a(z) = \frac{1}{1 - z} \left(\frac{1}{\delta} - \frac{1}{\lambda} \frac{1 - z\rho}{1 - z} \right). \tag{78}$$

Elimination of δ in eqs. (77) and (78) yields

$$a(z) = \frac{1}{\lambda} \frac{1}{1 - z} \left[\frac{z\rho}{1 - z} - j(z) \right], \tag{79}$$

which implies that the relation between j_n, a_n is

$$a_n = a_{n-1} + \frac{1}{\mu} - \frac{1}{\lambda} j_n, \tag{80}$$

$$a_n = \frac{n}{\mu} - \frac{1}{\lambda} \sum_{k=1}^{n} j_k. \tag{81}$$

This may be compared with eq. (48) for the $GI/M/1$.

From eq. (69), we have

$$\hat{K}_-(s) = \frac{\lambda}{\lambda - s} \hat{B}(\lambda), \tag{82}$$

$$\hat{K}_+(s) = \frac{\lambda}{\lambda - s} [\hat{B}(s) - \hat{B}(\lambda)], \tag{83}$$

$$\hat{K}_+(0) = 1 - \hat{B}(\lambda), \tag{84}$$

and

$$\hat{K}'_+(0) = \frac{1}{\lambda} [1 - \hat{B}(\lambda)] - \frac{1}{\mu}. \tag{85}$$

Thus, the recursion formulae eqs. (19) and (20) become, for $M/G/1$,

$$J_n = 1 - \hat{B}(\lambda) + J_{n-1} \frac{\hat{B}(\lambda)}{1 + \frac{1}{\lambda} g_{n-1}}, \tag{86}$$

$$\alpha_n = \left[1 - \hat{B}(\lambda) + \frac{\hat{B}(\lambda)}{1 + \frac{1}{\lambda} g_{n-1}} \right] \alpha_{n-1} + \frac{1}{\mu} - \frac{1}{\lambda} [1 - \hat{B}(\lambda)]. \tag{87}$$

Insofar as approximations imitate characteristics of an original, we may better apply the approximations. We will now show that the approximations J_n, α_n, satisfy eq. (80). From eq. (86) we find that

$$\frac{\hat{B}(\lambda)}{1 + \frac{1}{\lambda} g_{n-1}} = \frac{J_n - 1 + \hat{B}(\lambda)}{J_{n-1}}, \tag{88}$$

$$\alpha_{n-1} = \frac{1}{\lambda} \frac{J_{n-1}[J_n - 1 + \hat{B}(\lambda)]}{J_{n-1}\hat{B}(\lambda) - J_n + 1 - \hat{B}(\lambda)}, \tag{89}$$

and, from eq. (87),

$$\alpha_n - \alpha_{n-1} = -\frac{J_{n-1}\hat{B}(\lambda) - J_n + 1 - \hat{B}(\lambda)}{J_{n-1}} \cdot \alpha_{n-1} + \frac{1}{\mu} - \frac{1}{\lambda} [1 - \hat{B}(\lambda)]. \tag{90}$$

Thus, substitution of α_{n-1} from eq. (89) into the dexter of eq. (90) yields

$$\alpha_n = \alpha_{n-1} + \frac{1}{\mu} - \frac{1}{\lambda} J_n, \tag{91}$$

and

$$\alpha_n = \alpha_0 + \frac{n}{\mu} - \frac{1}{\lambda} \sum_{k=0}^{n} J_k. \tag{92}$$

From eqs. (81) and (92), we have

$$a_n - \alpha_n = -\alpha_0 + J_0 - \frac{1}{\lambda} \sum_{k=1}^{n} (j_k - J_k); \tag{93}$$

hence, for a stable queue,

$$j_k - J_k \rightarrow 0, \qquad k \rightarrow \infty. \tag{94}$$

Since $j_k \rightarrow \rho$, then also $J_k \rightarrow \rho$, $k \rightarrow \infty$. Unlike $GI/M/1$, however, $\alpha \neq a$. The equality that occurs in $GI/M/1$ was to be expected since the waiting-time distribution was approximated by an exponential in a portion of the integration producing eq. (15), and the exact equilibrium distribution is, in fact, exponential. This does not occur in $M/G/1$, however, since the waiting time is not exponential. The value for a is simply

$$a = \frac{\lambda}{2} \frac{B''(0)}{1-\rho}, \tag{95}$$

while solution of the equilibrium form of eqs. (86) and (87) shows that

$$\alpha = \frac{1}{\mu} \frac{\rho - 1 + \hat{B}(\lambda)}{(1-\rho)[1-\hat{B}(\lambda)]}; \tag{96}$$

thus, $\alpha \simeq a$ for small λ.

For the first numerical example, consider

$$\hat{A}(s) = \frac{\rho}{\rho + s}, \qquad \hat{B}(s) = \frac{1}{\left(1 + \frac{1}{2} s\right)^2}; \tag{97}$$

then

$$J_n = \frac{\rho}{(2+\rho)^2} \left(4 + \rho + J_{n-1} \frac{4}{\rho + g_{n-1}}\right), \tag{98}$$

$$\alpha_n = \frac{\rho}{(2+\rho)^2} \left(4 + \rho + \frac{4}{\rho + g_{n-1}}\right) \alpha_{n-1} + 1 - \frac{1}{\rho}\left(1 - \frac{4}{(2+\rho)^2}\right). \tag{99}$$

We have

$$J_1 = \frac{\rho(4+\rho)}{(2+\rho)^2}, \qquad \alpha_1 = 1 - \frac{1}{\rho}\left[1 - \frac{4}{(2+\rho)^2}\right] \tag{100}$$

for the queue starting empty. Table VI presents some numerical values for $\rho = 0.5$.

As another example let us consider the following $M/D/1$:

$$\hat{A}(s) = \frac{1}{1+2s}, \qquad \hat{B}(s) = e^{-s}. \tag{101}$$

The recursion formulae are

$$J_n = 1 - e^{-1/2} + J_{n-1} \frac{e^{-1/2}}{1 + 2g_{n-1}}, \tag{102}$$

and

$$\alpha_n = \left(1 - e^{-1/2} + \frac{e^{-1/2}}{1 + 2g_{n-1}}\right) \alpha_{n-1} + 2e^{-1/2} - 1. \tag{103}$$

The values are summarized in Table VII below.

V. *GI/G/1* QUEUEING SYSTEM

For the next group of examples, the control will provide only equilibrium values. The following uses an interrupted Poisson arrival stream[4] defined by

$$\hat{A}(s) = \frac{3.004s + 0.913216}{s^2 + 4.308s + 0.913216}. \tag{104}$$

The service time distribution is given by

$$\hat{B}(s) = \frac{3 + 5s}{(1 + 2s)(3 + 2s)}. \tag{105}$$

Thus,

$$\hat{K}(s) = \frac{(0.913216 - 3.004s)(3 + 5s)}{(0.223586 - s)(4.084414 - s)(1 + 2s)(3 + 2s)}, \tag{106}$$

$$\hat{K}_{-}(s) = \frac{0.0516472}{0.223586 - s} + \frac{0.672764}{4.084414 - s}, \tag{107}$$

$$\hat{K}_{+}(s) = \frac{0.182021}{1 + 2s} + \frac{1.266801}{3 + 2s}, \tag{108}$$

and

$$\hat{K}_{+}(0) = 0.604288, \qquad \hat{K}'_{+}(0) = -0.645553. \tag{109}$$

The recursion formulae are

$$J_n = 0.604288 + J_{n-1}\left(\frac{0.0516472}{0.223586 + g_{n-1}} + \frac{0.672764}{4.084414 + g_{n-1}}\right), \tag{110}$$

and

$$\alpha_n = \left(0.604288 + \frac{0.0516472}{0.223586 + g_{n-1}} + \frac{0.672764}{4.084414 + g_{n-1}}\right) \cdot \alpha_{n-1} + 0.645553. \tag{111}$$

Table VI—$M/G/1$, $\rho = 0.5$.

n	J	α	j	a
1	0.3600	0.2800	0.3600	0.2800
2	0.4245	0.4310	0.4266	0.4269
3	0.4515	0.5280	0.4547	0.5174
4	0.4666	0.5948	0.4698	0.5777
5	0.4762	0.6423	0.4789	0.6198
∞	0.5000	0.7778	0.5000	0.7500

Table VII—$M/D/1$, $\rho = 0.5$.

n	J	α	j	a
1	0.3935	0.2131	0.3935	0.2130
2	0.4443	0.3244	0.4482	0.3166
3	0.4655	0.3933	0.4701	0.3764
4	0.4773	0.4387	0.4812	0.4140
5	0.4846	0.4694	0.4876	0.4388
∞	0.5000	0.5415	0.5000	0.5000

Table VIII—$GI/G/1$, interrupted Poisson, $\rho = 0.7$.

n	J	α
1	0.6043	0.6456
2	0.7123	1.1509
3	0.7498	1.5762
4	0.7703	1.9470
5	0.7842	2.2769
6	0.7947	2.5743
7	0.8031	2.8452
∞	0.8824	8.0160

Table IX—Approximate and exact waiting-time distributions.

x	$w(x) \simeq$	$w(x) =$
0	0.1176	0.1183
1	0.2190	0.2212
2	0.3014	0.3026
3	0.3728	0.3717
4	0.4363	0.4323
5	0.4933	0.4863
6	0.5448	0.5350
7	0.5911	0.5789
8	0.6330	0.6187
9	0.6706	0.6547
10	0.7045	0.6872

Table VIII presents the numerical results. The exact equilibrium values are $j = 0.8817$, $a = 8.547$.

We will use this example to illustrate eq. (23) for the approximate distribution function. Using the approximate equilibrium values J, α, as in Table VIII, we find that

$$w(x) \simeq 1 - 0.894602e^{-0.11008x} + 0.0439472e^{-0.306135x} - 0.0317493e^{-0.969454x}. \qquad (112)$$

The exact waiting-time distribution is

$$w(x) = 1 - 0.8420579e^{-0.0990363x} - 0.03963069e^{-0.8959713x}. \qquad (113)$$

The numerical comparison is given in Table IX.

As another $GI/G/1$ example, we consider

$$\hat{A}(s) = \frac{1}{(1+s)(1+2s)}, \qquad (114)$$

and

$$\hat{B}(s) = \tfrac{1}{2}e^{-.7s} + \tfrac{1}{2}e^{-1.3s}. \qquad (115)$$

Here the service consists of a step function with two values. The recursive equations are

$$J_n = 0.157825 + J_{n-1}\left(\frac{1.226734}{1 + 2g_{n-1}} - \frac{0.384559}{1 + g_{n-1}}\right), \tag{116}$$

and

$$\alpha_n = \left(0.157825 + \frac{1.226734}{1 + 2g_{n-1}} - \frac{0.384559}{1 + g_{n-1}}\right)\alpha_{n-1} + 0.068909. \tag{117}$$

We show the numerical values in Table X. The exact equilibrium values are $j = 0.1798$, $\alpha = 0.09266$.

Table X—$GI/G/1$, two-step service time distribution, $\rho = 1/3$.

n	J	α
1	0.1578	0.06891
2	0.1741	0.08688
3	0.1782	0.09278
4	0.1795	0.09485
5	0.1800	0.09558
∞	0.1802	0.0960

For the last example, a $D/G/1$ will be considered. Let

$$\hat{A}(s) = e^{-sT}, \qquad \hat{B}(s) = \frac{1}{\left(1 + \frac{1}{2}s\right)^2}; \tag{118}$$

then

$$\hat{K}(s) = \frac{e^{sT}}{\left(1 + \frac{1}{2}s\right)^2}, \qquad \hat{K}_+(s) = \frac{e^{-2T}}{\left(1 + \frac{1}{2}s\right)^2} + \frac{2Te^{-2T}}{1 + \frac{1}{2}s}, \tag{119}$$

and

$$\hat{K}_-(s) = \hat{K}(s) - \hat{K}_+(s). \tag{120}$$

The recursion equations are

$$J_n = e^{-2T}(1 + 2T) + J_{n-1}\hat{K}_-(-g_{n-1}), \tag{121}$$

and

$$\alpha_n = [e^{-2T}(1 + 2T) + \hat{K}_-(-g_{n-1})]\alpha_{n-1} + e^{-2T}(1 + T). \tag{122}$$

Tables XI through XIII show values for $T = 2$, $10/9$, and 0.5, respectively.

To obtain exact equilibrium values for the stable queues, define the roots $\delta_1(z)$, $\delta_2(z)$ by

$$1 - ½\delta_1(z) = \sqrt{z}e^{-1/2T\delta_1(z)}, \tag{123}$$

Table XI—$D/G/1$, $t = 2$, $\rho = 0.5$.

n	J	α
1	0.0916	0.05495
2	0.1085	0.07016
3	0.1136	0.07561
4	0.1154	0.07775
5	0.1162	0.07863
∞	0.1167	0.07927

Table XII—$D/G/1$, $T = 10/9$, $\rho = 0.9$.

n	J	α
1	0.3492	0.2288
2	0.4614	0.3822
3	0.5185	0.5025
4	0.5545	0.6032
5	0.5801	0.6906
∞	0.7590	2.0631

Table XIII—$D/G/1$, $T = 0.5$, $\rho = 2$.

n	J	α
1	0.7358	0.5518
2	0.8875	1.0716
3	0.9362	1.5822
4	0.9566	2.0893
5	0.9672	2.5945
10	0.9850	5.1101
20	0.9928	10.1252

$$\tfrac{1}{2}\delta_2(z) - 1 = \sqrt{z}e^{-1/2T\delta_2(z)}; \tag{124}$$

then

$$j(z) = \frac{1}{1-z}\left[1 - \frac{1}{4}\,\delta_1(z)\delta_2(z)\right], \tag{125}$$

and

$$a(z) = \frac{1}{1-z}\left[\delta_1(z)^{-1} + \delta_2(z)^{-1} - 1\right]. \tag{126}$$

Thus, the equilibrium values are

$$j = 1 - \tfrac{1}{4}\delta_1(1)\delta_2(1), \qquad a = \delta_1(1)^{-1} + \delta_2(1)^{-1} - 1; \tag{127}$$

and, hence, for $T = 2$, we have $j = 0.1164$, $a = 0.07842$, and for $T = 10/9$, $j = 0.7587$, $a = 1.9895$.

VI. SYNOPSIS

We list the recursions and definitions of symbols here for ready reference.

6.1 Definitions

$A(x)$: interarrival time distribution.
$B(x)$: service time distribution.
$W_n(x)$: waiting-time distribution of nth arrival.

$$K(x) = \int_{0^-}^{\infty} B(x+y)dA(y).$$

λ: mean arrival rate.
μ: mean service rate.

$$\rho = \lambda/\mu.$$

$$\hat{K}(s) = \int_{-\infty}^{\infty} e^{-sx}dK(x).$$

$$\hat{K}_+(s) = \int_{0_-}^{\infty} e^{-sx}dK(x).$$

$$\hat{K}_-(s) = \int_{-\infty}^{0_-} e^{-sx}dK(x).$$

$$\hat{A}(s) = \int_{0_-}^{\infty} e^{-sx}dA(x).$$

$$\hat{B}(s) = \int_{0_-}^{\infty} e^{-sx}dB(x).$$

$$\hat{W}_n(s) = \int_{0_-}^{\infty} e^{-sx}dW_n(x).$$

j_n: probability nth arrival sees server is busy.
J_n: approximate evaluation of j_n.
a_n: mean waiting time of nth arrival.
α_n: approximate evaluation of a_n.

$$g_n = J_n/\alpha_n.$$

All symbols without the subscript n designate equilibrium values.

6.2 Recursion formulae

6.2.1 General—Transient

$$J_n = \hat{K}_+(0) + J_{n-1}\hat{K}(-g_{n-1}),$$
$$\alpha_n = [\hat{K}_+(0) + \hat{K}_-(-g_{n-1})]\alpha_{n-1} - \hat{K}'_+(0),$$

and

$$\hat{W}_n(s) = \hat{W}_{n-1}(s)\hat{K}_+(s) + \hat{K}_-(0) - \frac{s}{s + g_{n-1}} J_{n-1}\hat{K}_-(-g_{n-1}).$$

6.2.2 General—Equilibrium

$$J = \hat{K}_+(0) + J\hat{K}_-(-g),$$
$$\alpha = [\hat{K}_+(0) + \hat{K}_-(-g)]\alpha - \hat{K}'_+(0),$$

and

$$\hat{w}(s) = \frac{1}{1 - \hat{K}_+(s)}\left[\hat{K}_-(0) - \frac{s}{s+g} J\hat{K}_-(-g)\right].$$

6.2.3 GI/M/1

$$J_n = \hat{A}(\mu) + \frac{J_{n-1}}{1 - \frac{1}{\mu} g_{n-1}}[\hat{A}(g_{n-1}) - \hat{A}(\mu)],$$

$$\alpha_n = \left\{ \hat{A}(\mu) + \frac{1}{1 - \frac{1}{\mu} g_{n-1}}[\hat{A}(g_{n-1}) - \hat{A}(\mu)] \right\} \alpha_{n-1} + \frac{1}{\mu}\hat{A}(\mu),$$

$$J = \hat{A}[\mu(1 - J)],$$

and

$$\alpha = \frac{1}{\mu}\frac{J}{1 - J}.$$

6.2.4 M/G/1

$$J_n = 1 - \hat{B}(\lambda) + J_{n-1}\frac{\hat{B}(\lambda)}{1 + \frac{1}{\lambda} g_{n-1}},$$

$$\alpha_n = \left[1 - \hat{B}(\lambda) + \frac{\hat{B}(\lambda)}{1 + \frac{1}{\lambda} g_{n-1}} \right]\alpha_{n-1} + \frac{1}{\mu} - \frac{1}{\lambda}[1 - \hat{B}(\lambda)],$$

$$J = j = \rho,$$
$$\alpha = \frac{1}{\mu}\frac{\rho - 1 + \hat{B}(\lambda)}{(1 - \rho)[1 - \hat{B}(\lambda)]},$$

and

$$a = \frac{\lambda}{2} \frac{B''(0)}{1 - \rho}.$$

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