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Matrix Analysis of Mildly Nonlinear, Multiple-Input, Multiple-Output Systems With Memory

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A matrix method of analysis is developed for mildly nonlinear, multiple-input, multiple-output systems with memory (e.g., nonlinear multiport networks and multichannel communication systems). The method is based on a Volterra-series representation whose kernels are two-dimensional matrices rather than multidimensional arrays. This is made possible through the use of the Kronecker product of matrices, which results in a compact formulation. The response of the aforementioned systems to multiple sinusoidal excitations is also studied. Moreover, formulas are given for various system operations (e.g., addition, cascading, inversion, and feedback), which can be used to describe a complex system as an interconnection of simple subsystems.

I. INTRODUCTION

Communication, control, and instrumentation systems employ components, such as amplifiers and mixers, which are inherently nonlinear. Even when the nonlinearities are mild, as is often the case, they can produce bothersome signal distortion that limits the system performance. The nonlinear components themselves, and the other linear components used in the system, are generally frequency-dependent, i.e., they have memory. Numerous studies are available in the literature for the analysis of mildly nonlinear systems with memory through the use of Volterra-series expansions.[1-21] The classic paper by Bedrosian and Rice,[7] the recent paper by Chua and Ng,[14] and the book by Weiner and Spina[17] cover that subject very thoroughly. Also, the paper by Gopal, Njakhla, Singhal, and Vlach[12] is interesting in that it evaluates the range of accuracy of the Volterra-series approach by comparing it with a nearly exact, but quite involved, method of analysis. The book[18] and paper[19] by Schetzen deal mainly with random inputs. The conditions for the existence of a Volterra-series representation have recently been studied rigorously by Sandberg.[21]

For the most part, the studies mentioned above are limited to systems with one input and one output, i.e., "scalar" systems. This scalar representation is usually not easily applicable to Multiple-Input, Multiple-Output (MIMO) systems. Such systems include, for example, nonlinear multiport networks, multichannel communication systems, and transmitting or receiving systems employing multibeam antennas. In principle, one can represent these systems by a set of dependent scalar Volterra equations. This was done, for example, in the papers by Narayanan[3] and by Bussgang, Ehrman, and Graham,[9] where node equations were used to analyze nonlinear, two-port network models of bipolar transistor amplifiers. This method of analysis is tractable only when the numbers of nodes and of nonlinear elements in the network are small. For example, when the above authors considered the analysis of two-stage transistor amplifiers, they were forced by the complexity of the cascade equations involved to assume that the interaction between the stages, i.e., the loading effect of one stage on the other, is linear. While this might have been a reasonable approximation in their particular case, it is not valid in general. A symbolic matrix inversion algorithm that simplifies the computational aspects of the nodal method of analysis was recently discussed by Thapar and Leon.[15,16]

To conveniently handle the problem of two-port networks, or to analyze nonlinear multiport networks in general, one needs to use a black-box representation of the network, as is usually done in linear networks. For example, consider a nonlinear, two-port network, which has two independent port variables (e.g., the port currents) and two dependent port variables (e.g., the port voltages). One should be able to express the latter variables in terms of the former (e.g., by a nonlinear impedance representation). Furthermore, one should be able to perform transformations among various network representations (e.g., from impedance to cascade parameters), and to carry out the computations involved in interconnecting several networks together to form a complex network (e.g., through cascading). The same operations are also needed in the analysis of other nonlinear MIMO systems.

The purpose of this paper is to develop a method for analyzing mildly nonlinear MIMO systems with memory. This method, which employs Volterra-series whose kernels are two-dimensional matrices, facilitates the systematic performance of various useful system operations, such as addition, cascading, inversion, and feedback. The application of the results of this study to the analysis of mildly nonlinear multiport networks will be the subject of a future paper.

Actually, Weiner and Naditch,[10] and Gopal, Nakhla, Singhal, and Vlach[12] used multidimensional arrays of Volterra kernels to represent nonlinear, two-port networks. The same was suggested by Chua and Ng[14] for extending their results to multiple-input systems. All of these analyses can also be generalized to multiport networks and other MIMO systems. The resulting notation is similar to the index notation discussed in the beginning of the next section and in Appendix A. This notation, though more natural in its initial formulation, turns out to be cumbersome when attempting to perform the aforementioned system operations.

II. REPRESENTATION OF NONLINEAR MEMORYLESS MIMO SYSTEMS

A nonlinear, memoryless scaler system is characterized by its instantaneous input-output transfer function. When this function is analytic, as is the usual case encountered in practice, it can be represented by the power-series expansion

$$w = P^{(1)}u + P^{(2)}u^2 + P^{(3)}u^3 + \cdots, \tag{1}$$

where $u = u(t)$ is the input, $w = w(t)$ is the output, and $P^{(k)}$, $k = 1, 2, 3, \cdots$, are system constants. The corresponding representation of a nonlinear, memoryless, MIMO system with n inputs, $u_j = u_j(t)$, $j = 1, 2, \cdots, n$, and m outputs, $w_i = w_i(t)$, $i = 1, 2, \cdots, m$, is

$$\begin{aligned} w_i = {} & \sum_{j_1=1}^{n} P^{(1)}_{ij_1} u_{j_1} + \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} P^{(2)}_{ij_1j_2} u_{j_1} u_{j_2} \\ & + \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \sum_{j_3=1}^{n} P^{(3)}_{ij_1j_2j_3} u_{j_1} u_{j_2} u_{j_3} + \cdots, \quad i = 1, 2, \cdots, m, \end{aligned} \tag{2}$$

where $P^{(k)}_{ij_1 \cdots j_k}$, $k = 1, 2, 3, \cdots$, are $(k + 1)$-dimensional $m \times n \times \cdots \times n$ arrays of system constants. The notation used in (2) will be referred to as the "index notation." It is similar to that used in Refs. 10 and 12, but the superscripts and subscripts are interchanged. We now proceed to represent (2) in the "matrix notation."

Let

$$\mathbf{u} = \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ \cdot \\ \cdot \\ u_n(t) \end{bmatrix}, \quad \mathbf{w} = \mathbf{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \cdot \\ \cdot \\ \cdot \\ w_m(t) \end{bmatrix} \tag{3}$$

be the $n \times 1$ and $m \times 1$ input and output vectors, respectively. The first (i.e., linear) term in (2) can be written as an ordinary product of matrices in the form $\mathbf{w} = \mathbf{P}^{(1)} \cdot \mathbf{u}$, where $\mathbf{P}^{(1)}$ is the $m \times n$ matrix $[P^{(1)}_{ij}]$. We will now show that the remaining terms in (2) can also be written in a matrix form through the use of the Kronecker product of matrices.[22–24] Appendix B defines this product and gives some of its useful properties. Actually, Harper and Rugh[11] employed the Kronecker product in conjunction with state variables to study factorable, scalar, nonlinear systems. Also, Brockett[25,26] used a reduced form of the Kronecker product (to be explained shortly) in the state-variable representation of scalar, time-varying, nonlinear systems that are linear in the control variable.

As is explained below, the elements of the $(k + 1)$-dimensional, $m \times n \times \cdots \times n$ arrays, $\{P^{(k)}_{ij_1\cdots j_k}\}$, can be reorganized to form two-dimensional, $m \times n^k$ matrices, $\{\mathbf{P}^{(k)}\}$, such that (2) can be written in the matrix form

$$\mathbf{w} = \mathbf{P}^{(1)}\cdot\mathbf{u} + \mathbf{P}^{(2)}\cdot(\mathbf{u}\times\mathbf{u}) + \mathbf{P}^{(3)}\cdot(\mathbf{u}\times\mathbf{u}\times\mathbf{u}) + \cdots, \tag{4}$$

where "×" is the Kronecker-product sign. As mentioned in Appendix B, we will employ *left* Kronecker products.

To understand (4), we note from (61) that the k-fold Kronecker product $\mathbf{u}\times\mathbf{u}\times\cdots\times\mathbf{u}$ results in an $n^k \times 1$ vector whose jth element is given by

$$[\mathbf{u}\times\mathbf{u}\times\cdots\times\mathbf{u}]_j = u_{j_1}u_{j_2}\cdots u_{j_k}, \tag{5}$$

where $j_1, j_2, \cdots, j_k$ are uniquely determined from

$$j = j_1 + n(j_2 - 1) + \cdots + n^{k-1}(j_k - 1). \tag{6}$$

Thus, to make (4) equivalent to (2), the i-j element of the $m \times n^k$ matrix $\mathbf{P}^{(k)}$ should be given by

$$[\mathbf{P}^{(k)}]_{ij} = P^{(k)}_{ij_1\cdots j_k}, \tag{7}$$

where j is given by (6).

For example, if $m = n = 2$, (5)–(7) give

$$\mathbf{u}\times\mathbf{u} = \begin{bmatrix} u_1u_1 \\ u_2u_1 \\ u_1u_2 \\ u_2u_2 \end{bmatrix}, \mathbf{u}\times\mathbf{u}\times\mathbf{u} = \begin{bmatrix} u_1u_1u_1 \\ u_2u_1u_1 \\ u_1u_2u_1 \\ u_2u_2u_1 \\ u_1u_1u_2 \\ u_2u_1u_2 \\ u_1u_2u_2 \\ u_2u_2u_2 \end{bmatrix}, \tag{8}$$

$$\mathbf{P}^{(2)} = \begin{bmatrix} P^{(2)}_{111}P^{(2)}_{121}P^{(2)}_{112}P^{(2)}_{122} \\ P^{(2)}_{211}P^{(2)}_{221}P^{(2)}_{212}P^{(2)}_{222} \end{bmatrix}, \tag{9}$$

$$\mathbf{P}^{(3)} = \begin{bmatrix} P^{(3)}_{1111}P^{(3)}_{1211}P^{(3)}_{1121}P^{(3)}_{1221}P^{(3)}_{1112}P^{(3)}_{1212}P^{(3)}_{1122}P^{(3)}_{1222} \\ P^{(3)}_{2111}P^{(3)}_{2211}P^{(3)}_{2121}P^{(3)}_{2221}P^{(3)}_{2112}P^{(3)}_{2212}P^{(3)}_{2122}P^{(3)}_{2222} \end{bmatrix}. \tag{10}$$

Note that each of the Kronecker-product vectors given in (8) has redundant entries. In Brockett's notation cited earlier, these entries would be removed. For example, the 4×1 vector, $\mathbf{u} \times \mathbf{u}$, would be replaced by the 3×1 vector $[u_1^2, u_1u_2, u_2^2]$, and the corresponding 2×4 matrix, $\mathbf{P}^{(2)}$, given in (9) would be reduced to a 2×3 matrix, etc. However, when the system has memory, no redundant entries occur, since, as can be seen for example from (11), one needs to evaluate Kronecker products of the form $\mathbf{u}(t_1) \times \mathbf{u}(t_2)$, etc., where $t_1 \neq t_2$.

In the remainder of the paper, we will employ the compact matrix notation used in (4) rather than the index notation used in (2). However, on some occasions, it is helpful to keep track of the interrelation between the two notations. Thus, some key equations in the paper are rewritten in Appendix A in the index notation.

III. REPRESENTATION OF NONLINEAR MIMO SYSTEMS WITH MEMORY

The usual Volterra-series expansion used to represent nonlinear, time-invariant, scalar systems with memory[1–21] can be generalized through the use of the notation of (4) to represent MIMO systems by the matrix equation

$$\begin{aligned}\mathbf{w}(t) = &\int_{-\infty}^{\infty} \mathbf{p}^{(1)}(\tau_1)\cdot\mathbf{u}(t-\tau_1)d\tau_1 \\ &+ \iint_{-\infty}^{\infty} \mathbf{p}^{(2)}(\tau_1, \tau_2)\cdot[\mathbf{u}(t-\tau_1)\times\mathbf{u}(t-\tau_2)]d\tau_1 d\tau_2 \\ &+ \iiint_{-\infty}^{\infty} \mathbf{p}^{(3)}(\tau_1, \tau_2, \tau_3) \\ &\qquad \cdot[\mathbf{u}(t-\tau_1)\times\mathbf{u}(t-\tau_2)\times\mathbf{u}(t-\tau_3)]d\tau_1 d\tau_2 d\tau_3 \\ &+ \cdots,\end{aligned} \tag{11}$$

where $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are, respectively, the $n \times 1$ and $m \times 1$ input and output vectors given by (3), and where $\mathbf{p}^{(k)}(\tau_1, \cdots, \tau_k)$, $k = 1, 2, 3, \cdots$, are two-dimensional, $m \times n^k$ matrices of system kernels. Note that if $\mathbf{p}^{(k)}(\tau_1, \cdots, \tau_k) = \mathbf{P}^{(k)}\delta(\tau_1) \cdots \delta(\tau_k)$, where $\delta(\tau)$ is the unit impulse function, then the system becomes memoryless, and (11) reduces to (4).

As is the case for linear systems, it is more convenient to represent (11) in the frequency domain. To do this, we introduce the dummy time variables, $t_1, t_2, \cdots, t_k$, and rewrite the kth order output component in (11) as

$$\mathbf{w}^{(k)}(t_1, \cdots, t_k) = \int_{-\infty}^{\infty} \cdots \int \mathbf{p}^{(k)}(\tau_1, \cdots, \tau_k)$$

$$\cdot[\mathbf{u}(t_1 - \tau_1) \times \cdots \times \mathbf{u}(t_k - \tau_k)]d\tau_1 \cdots d\tau_k. \quad (12)^\dagger$$

Thus, (11) becomes

$$\mathbf{w}(t) = \mathbf{w}^{(1)}(t) + \mathbf{w}^{(2)}(t, t) + \mathbf{w}^{(3)}(t, t, t) + \cdots . \quad (13)$$

Now, we introduce the single-dimensional Fourier-transform pair

$$\mathbf{X}(f) = \int_{-\infty}^{\infty} \mathbf{x}(t)\exp(-j2\pi ft)dt, \quad (14a)$$

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} \mathbf{X}(f)\exp(j2\pi ft)df, \quad (14b)$$

to represent the transformations $\mathbf{u}(t) \leftrightarrow \mathbf{U}(f)$ and $\mathbf{w}(t) \leftrightarrow \mathbf{W}(f)$. Similarly, we introduce the multi-dimensional Fourier-transform pair

$$\mathbf{Y}(f_1, \cdots, f_k) = \int_{-\infty}^{\infty} \cdots \int \mathbf{y}(t_1, \cdots, t_k)$$

$$\cdot\exp[-j2\pi(f_1t_1 + \cdots + f_kt_k)]dt_1 \cdots dt_k, \quad (15a)$$

$$\mathbf{y}(t_1, \cdots, t_k) = \int_{-\infty}^{\infty} \cdots \int \mathbf{Y}(f_1, \cdots, f_k)$$

$$\cdot\exp[j2\pi(f_1t_1 + \cdots + f_kt_k)]df_1 \cdots df_k, \quad (15b)$$

to represent the transformations $\mathbf{p}^{(k)}(\tau_1, \cdots, \tau_k) \leftrightarrow \mathbf{P}^{(k)}(f_1, \cdots, f_k)$ and $\mathbf{w}^{(k)}(t_1, \cdots, t_k) \leftrightarrow \mathbf{W}^{(k)}(f_1, \cdots, f_k)$. It can be shown from (14) and (15) that (12) can be written in the frequency domain as (see Refs. 1 through 4, 7 through 9, 14 and 17)

$$\mathbf{W}^{(k)}(f_1, \cdots, f_k) = \mathbf{P}^{(k)}(f_1, \cdots, f_k)\cdot[\mathbf{U}(f_1) \times \cdots \times \mathbf{U}(f_k)]. \quad (16)^\dagger$$

The Fourier transform of the output becomes

$$\mathbf{W}(f) = \mathbf{W}^{(1)}(f) + \int_{-\infty}^{\infty} \mathbf{W}^{(2)}(f_1, f - f_1)df_1$$

$$+ \int_{-\infty}^{\infty}\int \mathbf{W}^{(3)}(f_1, f_2, f - f_1 - f_2)df_1df_2 + \cdots . \quad (17)$$

† All equations in the paper marked by a dagger are rewritten in the index notation in Appendix A, where the same equation numbers are used.

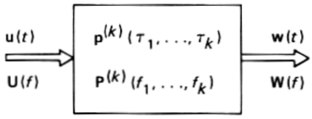

Fig. 1—A nonlinear, time-invariant, MIMO system with memory having n inputs and m outputs.

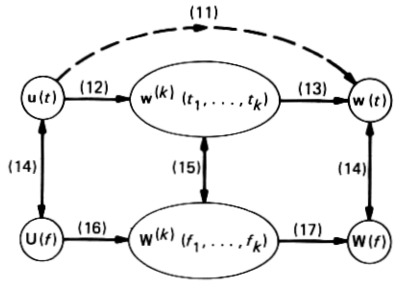

Fig. 2—Interrelations among the input, output components, and total output in the time and frequency domains for the nonlinear MIMO system of Fig. 1. The numbers in parentheses represent equation numbers in the text.

Note from (13) and (17) that the single-dimensional Fourier transform of $\mathbf{w}^{(k)}(t, \cdots, t)$ is given by the kth term in (17), which is *not* equal to $\mathbf{W}^{(k)}(f, \cdots, f)$ unless $k = 1$.

A schematic diagram of the system represented by (11) through (17) is given in Fig. 1. The interrelations among the input, the output components, and the total output in the time and frequency domains, and the corresponding equation numbers, are indicated in the flow-chart of Fig. 2.

IV. KERNEL SYMMETRIZATION

The representation of the response of a nonlinear scalar system to sinusoidal and Gaussian excitations is greatly simplified if each of the kernels, $P^{(k)} f_1, \cdots, f_k)$, or equivalently, $p^{(k)}(\tau_1, \cdots, \tau_k)$, is a symmetric function of its arguments.[7-9,14] The generalization of this symmetry requirement to nonlinear MIMO systems is somewhat more involved. Following the reasoning given in the aforementioned references, one can show that it is the output components, $\mathbf{W}^{(k)}(f_1, \cdots, f_k)$ given by (16), or equivalently, $\mathbf{w}^{(k)}(t_1, \cdots, t_k)$ given by (12), that are required to be symmetric functions of their arguments. For example, for $k = 2$, it is required that $\mathbf{W}^{(2)}(f_1, f_2) = \mathbf{W}^{(2)}(f_2, f_1)$; and thus, from (16),

$$\mathbf{P}^{(2)}(f_1, f_2)\cdot(\mathbf{u}_1 \times \mathbf{u}_2) = \mathbf{P}^{(2)}(f_2, f_1)\cdot(\mathbf{u}_2 \times \mathbf{u}_1), \tag{18a}$$

where $\mathbf{U}(f_i)$ is replaced by $\mathbf{u}_i$ for generality. Similarly, for $k = 3$, it is required that $\mathbf{W}^{(3)}(f_1, f_2, f_3) = \mathbf{W}^{(3)}(f_\alpha, f_\beta, f_\gamma)$; and thus, from (16),

$$\mathbf{P}^{(3)}(f_1, f_2, f_3)\cdot(\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3) = \mathbf{P}^{(3)}(f_\alpha, f_\beta, f_\gamma)\cdot(\mathbf{u}_\alpha \times \mathbf{u}_\beta \times \mathbf{u}_\gamma), \tag{18b}$$

where α, β, γ assume all permutations of *1, 2, 3.* For a scalar system, (18a) and (18b) are indeed equivalent to requiring the corresponding system kernels to be symmetric functions of their arguments, as mentioned above.

To find the symmetry requirement implied by (18) on the kernels of a MIMO system, we need to introduce the $n^2 \times n^2$ "reversing" matrix, $\mathbf{R}$, and the six $n^3 \times n^3$ "permutation" matrices, $\mathbf{\Phi}_{\alpha\beta\gamma}$, where α, β, γ assume all permutations of *1, 2, 3.* These matrices have properties such that if $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are $n \times 1$ vectors, then

$$\mathbf{R}\cdot(\mathbf{u}_1 \times \mathbf{u}_2) = \mathbf{u}_2 \times \mathbf{u}_1, \tag{19}$$

$$\mathbf{\Phi}_{\alpha\beta\gamma}\cdot(\mathbf{u}_1 \times \mathbf{u}_2 \times \mathbf{u}_3) = \mathbf{u}_\alpha \times \mathbf{u}_\beta \times \mathbf{u}_\gamma. \tag{20}$$

Appendix C defines these matrices and gives some of their useful properties.

Finally, (18) through (20) give the required symmetry conditions of the kernels as

$$\mathbf{P}^{(2)}(f_1, f_2) = \mathbf{P}^{(2)}(f_2, f_1)\cdot\mathbf{R}, \qquad (21a)^\dagger$$

$$\mathbf{P}^{(3)}(f_1, f_2, f_3) = \mathbf{P}^{(3)}(f_\alpha, f_\beta, f_\gamma)\cdot\mathbf{\Phi}_{\alpha\beta\gamma}. \qquad (21b)^\dagger$$

The generalization of (21) to higher-order kernels requires the introduction of permutation matrices of more than three indices.

If the given system kernels, say, $\hat{\mathbf{P}}^{(2)}(f_1, f_2)$ and $\hat{\mathbf{P}}^{(3)}(f_1, f_2, f_3)$, are unsymmetric, they can be symmetrized, i.e., made to satisfy (21), through the use of the relations

$$\mathbf{P}^{(2)}(f_1, f_2) = \tfrac{1}{2}\,[\hat{\mathbf{P}}^{(2)}(f_1, f_2) + \hat{\mathbf{P}}^{(2)}(f_2, f_1)\cdot\mathbf{R}], \qquad (22a)^\dagger$$

$$\mathbf{P}^{(3)}(f_1, f_2, f_3) = \tfrac{1}{6}\sum_{\alpha,\beta,\gamma} \hat{\mathbf{P}}^{(3)}(f_\alpha, f_\beta, f_\gamma)\cdot\mathbf{\Phi}_{\alpha\beta\gamma}, \qquad (22b)^\dagger$$

where the summation is performed over α, β, γ assuming all *6* permutations of *1, 2, 3.* These symmetrization relations are generalizations of those discussed in Refs. 7, 9, and 14 for scalar kernels.

V. RESPONSE TO SINUSOIDAL EXCITATION

The response of a nonlinear scalar system to multiple-sinusoidal excitation has been studied by several authors including Bedrosian and Rice,[7] Goldman,[8] and Chua and Ng.[14] Here we generalize some of their results to nonlinear MIMO systems.

5.1 Multiple-exponential excitation

Let the input vector be

$$\mathbf{u}(t) = \sum_{i=1}^{l} \mathbf{u}_i \exp(j2\pi f_i t), \tag{23}$$

where the $\mathbf{u}_i$'s are time-independent, complex, $n \times 1$ vectors. The Fourier transform of $\mathbf{u}(t)$ is

$$\mathbf{U}(f) = \sum_{i=1}^{l} \mathbf{u}_i \delta(f - f_i). \tag{24}$$

Substituting (24) into (16), and using (15b), one obtains the kth order output component

$$w^{(k)}(t) \equiv \mathbf{w}^{(k)}(t, \cdots, t) = \sum_{i_1=1}^{l} \cdots \sum_{i_k=1}^{l} \{[\mathbf{P}^{(k)}(f_{i_1}, \cdots, f_{i_k}) \cdot (\mathbf{u}_{i_1} \times \cdots \times \mathbf{u}_{i_k})]\exp[j2\pi(f_{i_1} + \cdots + f_{i_k})t]\}. \tag{25}$$

Finally, the output, $\mathbf{w}(t)$, is obtained from (13), i.e., by summing $\mathbf{w}^{(k)}(t)$ from $k = 1$ up to any desired order. Note that (25) is valid whether or not the system kernels are symmetric.

5.2 Single-frequency excitation

Let the (real) $n \times 1$ input vector be

$$\begin{aligned}\mathbf{u}(t) &= \text{Real}[\mathbf{a} \exp(j2\pi ft)] \\ &= \tfrac{1}{2}\,\mathbf{a} \exp(j2\pi ft) + \tfrac{1}{2}\mathbf{a}^*\exp(-j2\pi ft),\end{aligned} \tag{26}$$

where the asterisk refers to complex conjugation. Comparing (26) to (23), one obtains $l = 2$, $\mathbf{u}_1 = \frac{1}{2}\,\mathbf{a}$, $\mathbf{u}_2 = \frac{1}{2}\,\mathbf{a}^*$, $f_1 = f$, and $f_2 = -f$. Thus, using (25), and assuming that the kernels are symmetric, i.e., that (18) is satisfied, one obtains the following expressions for the various kth order output components:

$$\mathbf{w}^{(1)}(t) = \tfrac{1}{2}\,[\mathbf{P}^{(1)}(f) \cdot \mathbf{a}]\exp(j2\pi ft) + \tfrac{1}{2}\,[\mathbf{P}^{(1)}(-f) \cdot \mathbf{a}^*]\exp(-j2\pi ft). \tag{27a}$$

$$\begin{aligned}\mathbf{w}^{(2)}(t) &= \tfrac{1}{2}\,[\mathbf{P}^{(2)}(f, -f) \cdot (\mathbf{a} \times \mathbf{a}^*)] \leftarrow (d-c \text{ term}) \\ &+ \tfrac{1}{4}\,[\mathbf{P}^{(2)}(f, f) \cdot (\mathbf{a} \times \mathbf{a})]\exp[j2\pi(2f)t] \\ &+ \tfrac{1}{4}[\mathbf{P}^{(2)}(-f, -f) \cdot (\mathbf{a}^* \times \mathbf{a}^*)]\exp[-j2\pi(2f)t].\end{aligned} \tag{27b}$$

$$\begin{aligned}\mathbf{w}^{(3)}(t) &= \tfrac{3}{8}\,[\mathbf{P}^{(3)}(f, f, -f) \cdot (\mathbf{a} \times \mathbf{a} \times \mathbf{a}^*)]\exp(j2\pi ft) \\ &+ \tfrac{3}{8}\,[\mathbf{P}^{(3)}(-f, -f, f) \cdot (\mathbf{a}^* \times \mathbf{a}^* \times \mathbf{a})]\exp(-j2\pi ft) \\ &+ \tfrac{1}{8}\,[\mathbf{P}^{(3)}(f, f, f) \cdot (\mathbf{a} \times \mathbf{a} \times \mathbf{a})]\exp[j2\pi(3f)t] \\ &+ \tfrac{1}{8}\,[\mathbf{P}^{(3)}(-f, -f, -f) \cdot (\mathbf{a}^* \times \mathbf{a}^* \times \mathbf{a}^*)]\exp[-j2\pi(3f)t].\end{aligned} \tag{27c}$$

Note that the asterisks on the **a**'s correspond in number and location to the negative signs in the frequency arguments of the associated kernels.

If the system is real, i.e., if $\mathbf{p}^{(k)}(\tau_1, \cdots, \tau_k)$, $k = 1, 2, 3, \cdots$, are real, then it can be shown from (15) that

$$\mathbf{P}^{(k)}(f_1, \cdots, f_k) = [\mathbf{P}^{(k)}(-f_1, \cdots, -f_k)]^*. \tag{28}$$

As expected, (28) implies that all the output components given in (27) are real. In that case, it can be shown through generalizing (27) that the total mth harmonic output term, $\mathbf{w}_m(t)$, $m = 0, 1, 2, \cdots$, is given by

$$\mathbf{w}_m(t) = \epsilon_m \text{Real}\left[\exp(j2\pi m f) \sum_{k=m,m+2,\cdots}\left\{2^{-k}\binom{k}{\frac{k-m}{2}} \cdot \mathbf{P}^{(k)}(\underbrace{f, \cdots, f}_{(k+m)/2}, \underbrace{-f, \cdots, -f}_{(k-m)/2}) \cdot [\mathbf{a}^{[(k+m)/2]} \times (\mathbf{a}^*)^{[(k-m)/2]}]\right\}\right], \tag{29}$$

where $\mathbf{a}^{[l]}$ is the l-fold Kronecker product $\mathbf{a} \times \cdots \times \mathbf{a}$, ϵ_m is the Neuman factor (which is equal to 1 when $m = 0$, and is equal to 2 when $m \neq 0$), and $\mathbf{P}^{(0)}$ is defined to be zero.

Because of the symmetry conditions of (21), the kernels used in (27) and (29) satisfy the relations

$$\mathbf{P}^{(2)}(f, f) = \mathbf{P}^{(2)}(f, f) \cdot \mathbf{R}, \qquad (30a)^\dagger$$

$$\mathbf{P}^{(2)}(f, -f) = [\mathbf{P}^{(2)}(f, -f)]^* \cdot \mathbf{R}, \qquad (30b)^\dagger$$

$$\mathbf{P}^{(3)}(f, f, -f) = \mathbf{P}^{(3)}(f, f, -f) \cdot \mathbf{\Phi}_{213}, \qquad (30c)^\dagger$$

$$\mathbf{P}^{(3)}(f, f, f) = \mathbf{P}^{(3)}(f, f, f) \cdot \mathbf{\Phi}_{\alpha\beta\gamma}. \qquad (30d)^\dagger$$

In addition to the kernel symmetry requirement, (30b) is based on the assumption that the system is real, i.e., that (28) is satisfied. The implication of (30) is that the elements of each of the system kernels are not all independent. For example, if $n = 2$, (30a) through (30d) imply, respectively, that (*i*) columns 2 and 3 of $\mathbf{P}^{(2)}(f, f)$ are equal; (*ii*) column 2 of $\mathbf{P}^{(2)}(f, -f)$ is the complex conjugate of column 3, and columns 1 and 4 are real; (*iii*) columns 2 and 3 of $\mathbf{P}^{(3)}(f, f, -f)$ are equal, and so are columns 6 and 7; and (*iv*) columns 2, 3, and 5 of $\mathbf{P}^{(3)}(f, f, f)$ are equal, and so are columns 4, 6, and 7. It is worth mentioning that (30a) and (30d), respectively, would also be satisfied by $\mathbf{P}^{(2)}$ and $\mathbf{P}^{(3)}$ of the memoryless system represented by (4).

5.3 Two-frequency excitation

Let the (real) $n \times 1$ input vector be

$$\mathbf{u}(t) = \text{Real}[\mathbf{a}\exp(j2\pi f_a t) + \mathbf{b}\exp(j2\pi f_b t)]. \tag{31}$$

We assume that the system is real, and that the kernels are symmetric, i.e., that (28) and (18) are satisfied. One can use (25) to obtain the output corresponding to (32) by following the same steps used to derive (29). The leading terms at some of the various output frequencies are:

$$\mathbf{w}(t)|_{d-c} \approx \tfrac{1}{2}[\mathbf{P}^{(2)}(f_a, -f_a)\cdot(\mathbf{a} \times \mathbf{a}^*) + \mathbf{P}^{(2)}(f_b, -f_b)\cdot(\mathbf{b} \times \mathbf{b}^*)]. \quad (32a)$$

$$\begin{aligned}\mathbf{w}(t)|_{f_a} \approx \text{Real}\{&\exp(j2\pi f_a t)[\mathbf{P}^{(1)}(f_a)\cdot\mathbf{a} \\ &+ \tfrac{3}{4}\mathbf{P}^{(3)}(f_a, f_a, -f_a)\cdot(\mathbf{a} \times \mathbf{a} \times \mathbf{a}^*) \\ &+ \tfrac{3}{2}\mathbf{P}^{(3)}(f_a, f_b, -f_b)\cdot(\mathbf{a} \times \mathbf{b} \times \mathbf{b}^*)]\}. \end{aligned} \quad (32b)$$

$$\begin{aligned}\mathbf{w}(t)|_{2f_a-f_b} \approx \text{Real}\{&\tfrac{3}{4}\exp[j2\pi(2f_a - f_b)t] \\ &\cdot\mathbf{P}^{(3)}(f_a, f_a, -f_b)\cdot(\mathbf{a} \times \mathbf{a} \times \mathbf{b}^*)\}. \end{aligned} \quad (32c)$$

$$\begin{aligned}\mathbf{w}(t)|_{lf_a \pm mf_b} \approx \text{Real}\Bigg\{&2^{-(l+m-1)}\binom{l+m}{l}\exp[j2\pi(lf_a \pm mf_b)t] \\ &\cdot\mathbf{P}^{(l+m)}(\underbrace{f_a, \cdots, f_a}_{l}, \underbrace{\pm f_b, \cdots, \pm f_b}_{m})\cdot[\mathbf{a}^{[l]} \times (\mathbf{b}^{\pm})^{[m]}]\Bigg\}, \end{aligned} \quad (32d)$$

where $l > 0$ and $m \geq 0$, and where we defined $\mathbf{b}^+ \equiv \mathbf{b}$ and $\mathbf{b}^- \equiv \mathbf{b}^*$.

5.4 Three-frequency excitation

Let the (real) $n \times 1$ input vector be

$$\mathbf{u}(t) = \text{Real}[\mathbf{a}\exp(j2\pi f_a t) + \mathbf{b}\exp(j2\pi f_b t) + \mathbf{c}\exp(j2\pi f_c t)]. \quad (33)$$

Again, we assume that the system is real, and that the kernels are symmetric. Following the same steps used to derive (29) and (32), one can obtain the following leading terms at some of the various output frequencies:

$$\begin{aligned}\mathbf{w}(t)|_{d-c} \approx \tfrac{1}{2}[&\mathbf{P}^{(2)}(f_a, -f_a)\cdot(\mathbf{a} \times \mathbf{a}^*) + \mathbf{P}^{(2)}(f_b, -f_b) \\ &\cdot(\mathbf{b} \times \mathbf{b}^*) + \mathbf{P}^{(2)}(f_c, -f_c)\cdot(\mathbf{c} \times \mathbf{c}^*)]. \end{aligned} \quad (34a)$$

$$\begin{aligned}\mathbf{w}(t)|_{f_a} \approx \text{Real}\{&\exp(j2\pi f_a t)[\mathbf{P}^{(1)}(f_a)\cdot\mathbf{a} + \tfrac{3}{4}\mathbf{P}^{(3)}(f_a, f_a, -f_a) \\ &\cdot(\mathbf{a} \times \mathbf{a} \times \mathbf{a}^*) + \tfrac{3}{2}\mathbf{P}^{(3)}(f_a, f_b, -f_b)\cdot(\mathbf{a} \times \mathbf{b} \times \mathbf{b}^*) \\ &+ \tfrac{3}{2}\mathbf{P}^{(3)}(f_a, f_c, -f_c)\cdot(\mathbf{a} \times \mathbf{c} \times \mathbf{c}^*)]\}. \end{aligned} \quad (34b)$$

$$\begin{aligned}\mathbf{w}(t)|_{f_a+f_b-f_c} \approx \text{Real}\{&\tfrac{3}{2}\exp[j2\pi(f_a + f_b - f_c)t] \\ &\cdot\mathbf{P}^{(3)}(f_a, f_b, -f_c)\cdot(\mathbf{a} \times \mathbf{b} \times \mathbf{c}^*)\}. \end{aligned} \quad (34c)$$

$$\mathbf{w}(t)|_{kf_a \pm lf_b \oplus mf_c}$$
$$\approx \text{Real}\left\{2^{-(k+l+m-1)}\frac{(k+l+m)!}{k!l!m!}\exp[j2\pi(kf_a \pm lf_b \oplus mf_c)t]\right.$$
$$\cdot\mathbf{P}^{(k+l+m)}(\underbrace{f_a, \cdots, f_a}_{k}, \underbrace{\pm f_b, \cdots, \pm f_b}_{l}, \underbrace{\oplus f_c, \cdots, \oplus f_c}_{m})$$
$$\left.\cdot[\mathbf{a}^{[k]} \times (\mathbf{b}^{\pm})^{[l]} \times (\mathbf{c}^{\oplus})^{[m]}]\right\}, \tag{34d}$$

where k, l, $m \geq 0$, but at least one of them being nonzero, and where the sign symbols $\pm$ and $\oplus$ are each consistent throughout the equation, but are otherwise independent.

VI. SYSTEM OPERATIONS

6.1 Operational notation

Let the input-output relations given in (11) through (17) be written symbolically as

$$\mathbf{W}_m = \{\mathbf{P}_{m,n}^{(k)}\} o \mathbf{U}_n, \tag{35}$$

where "o" means "operating on." The frquency dependence has been omitted for simplicity. The subscripts n and m are included to emphasize the numbers of inputs and outputs. On some occasions, these subscripts will be eliminated.

If the system is linear, i.e., if $\mathbf{P}^{(k)} = \mathbf{0}$ for $k > 1$, the operation in (35) reduces to an ordinary matrix product. Thus,

$$\mathbf{W} = \{\mathbf{P}^{(1)}\} o \mathbf{U} = \mathbf{P}^{(1)} \cdot \mathbf{U}. \tag{36}$$

The operational notation of (35), and the three system operations of addition, cascading, and inversion, which are discussed in the next three subsections, form an algebraic structure that permits a shorthand description of complex interconnections of nonlinear MIMO systems. The laws of this algebra[1] are identical to the algebra of linear systems (i.e., the algebra of matrices) with two important exceptions—the left distributive law does not hold, and the laws of multiplication by a scalar constant are more complex.

6.2 Addition

Two systems, $\{\mathbf{P}_{m,n}^{(k)}\}$ and $\{\mathbf{Q}_{m,n}^{(k)}\}$, having the same number of inputs, n, and the same number of outputs, m, are said to be "added" if they share the same input vector, $\mathbf{U}_n$, and if their respective outputs are added to form the final output vector, $\mathbf{W}_m$. This operation, which is shown schematically in Fig. 3, is represented by

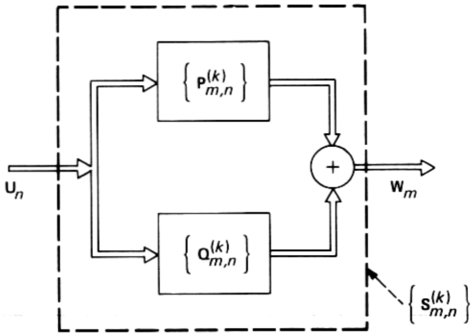

Fig. 3—A schematic representation of the addition operation $\{\mathbf{S}^{(k)}_{m,n}\} = \{\mathbf{P}^{(k)}_{m,n}\} + \{\mathbf{Q}^{(k)}_{m,n}\}$.

$$\begin{aligned}\mathbf{W}_m &= \{\mathbf{P}^{(k)}_{m,n}\}o\mathbf{U}_n + \{\mathbf{Q}^{(k)}_{m,n}\}o\mathbf{U}_n \\ &= [\{\mathbf{P}^{(k)}_{m,n}\} + \{\mathbf{Q}^{(k)}_{m,n}\}]o\mathbf{U}_n \\ &= \{\mathbf{S}^{(k)}_{m,n}\}o\mathbf{U}_n. \end{aligned} \tag{37}$$

The kernels of the sum system,

$$\{\mathbf{S}^{(k)}_{m,n}\} = \{\mathbf{P}^{(k)}_{m,n}\} + \{\mathbf{Q}^{(k)}_{m,n}\}, \tag{38}$$

are given by

$$\mathbf{S}^{(k)}(f_1, \cdots, f_k) = \mathbf{P}^{(k)}(f_1, \cdots, f_k) + \mathbf{Q}^{(k)}(f_1, \cdots, f_k), \tag{39}$$

where the plus sign refers to matrix addition.

One can define a subtraction operation in an obvious manner. A multiplication operation,[1,14] which is more involved, can also be defined.

6.3 Cascading

When the output vector, $\mathbf{W}_m$, of a system, $\{\mathbf{P}^{(k)}_{m,n}\}$, is used as an input vector to a second system, $\{\mathbf{Q}^{(k)}_{l,m}\}$, whose output vector is $\mathbf{X}_l$, the two systems are said to be in "cascade." This operation, which is shown schematically in Fig. 4, is represented by

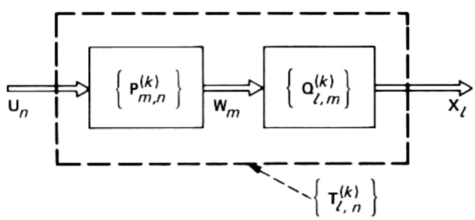

Fig. 4—A schematic representation of the cascade operation $\{\mathbf{T}^{(k)}_{l,n}\} = \{\mathbf{Q}^{(k)}_{l,m}\} * \{\mathbf{P}^{(k)}_{m,n}\}$.

$$\begin{aligned}\mathbf{X}_l &= \{\mathbf{Q}_{l,m}^{(k)}\} o \mathbf{W}_m \\ &= \{\mathbf{Q}_{l,m}^{(k)}\} o [\{\mathbf{P}_{m,n}^{(k)}\} o \mathbf{U}_n] \\ &= [\{\mathbf{Q}_{l,m}^{(k)}\} * \{\mathbf{P}_{m,n}^{(k)}\}] o \mathbf{U}_n \\ &= \{\mathbf{T}_{l,n}^{(k)}\} o \mathbf{U}_n, \end{aligned} \tag{40}$$

where the asterisk refers to the cascade operation. The kernels of the cascade system,

$$\{\mathbf{T}_{l,n}^{(k)}\} = \{\mathbf{Q}_{l,m}^{(k)}\} * \{\mathbf{P}_{m,n}^{(k)}\}, \tag{41}$$

can be obtained by substituting the output expression of the first system into the system equations of the second system, as was done in Refs. 3 and 9 to derive the cascade relations of scalar systems. This procedure is straightforward, but somewhat tedious. A simpler approach is to employ the harmonic probing method discussed in Refs. 7 and 9, and the expression for the response of nonlinear vector systems to multiple-exponential excitation given in (25). The resulting relations for the cascade kernels are

$$\mathbf{T}^{(1)}(f_1) = \mathbf{Q}^{(1)}(f_1) \cdot \mathbf{P}^{(1)}(f_1), \tag{42a†}$$

$$\mathbf{T}^{(2)}(f_1, f_2) = \mathbf{Q}^{(1)}(f_1 + f_2) \cdot \mathbf{P}^{(2)}(f_1, f_2) + \mathbf{Q}^{(2)}(f_1, f_2) \cdot [\mathbf{P}^{(1)}(f_1) \times \mathbf{P}^{(1)}(f_2)], \tag{42b†}$$

$$\begin{aligned}\hat{\mathbf{T}}^{(3)}(f_1, f_2, f_3) &= \mathbf{Q}^{(1)}(f_1 + f_2 + f_3) \cdot \mathbf{P}^{(3)}(f_1, f_2, f_3) \\ &+ \mathbf{Q}^{(2)}(f_1, f_2 + f_3) \cdot [\mathbf{P}^{(1)}(f_1) \times \mathbf{P}^{(2)}(f_2, f_3)] \\ &+ \mathbf{Q}^{(2)}(f_1 + f_2, f_3) \cdot [\mathbf{P}^{(2)}(f_1, f_2) \times \mathbf{P}^{(1)}(f_3)] \\ &+ \mathbf{Q}^{(3)}(f_1, f_2, f_3) \cdot [\mathbf{P}^{(1)}(f_1) \times \mathbf{P}^{(1)}(f_2) \times \mathbf{P}^{(1)}(f_3)]. \end{aligned} \tag{42$\hat{c}$†}$$

A generalization of (42) for arbitrary k is given in Appendix D.

If the kernels of the cascaded systems are symmetric, i.e., satisfy (21), then it can be shown that the resulting second-order kernel given by (42b) is also symmetric. However, the resulting third-order kernel given by (42ĉ) is not symmetric. This fact is indicated by the presence of the circumflexes.

As mentioned in Section IV, it is desirable to deal with symmetric kernels. Thus, using the symmetrization relation given in (22b), assuming that the kernels of the cascaded systems are symmetric, and employing the properties of the reversing and permutation matrices given in Appendix C, one obtains the symmetric form of (42ĉ) as

$$\begin{aligned}\mathbf{T}^{(3)}(f_1, f_2, f_3) &= \mathbf{Q}^{(1)}(f_1 + f_2 + f_3) \cdot \mathbf{P}^{(3)}(f_1, f_2, f_3) \\ &+ \tfrac{2}{3}\{\mathbf{Q}^{(2)}(f_1, f_2 + f_3) \cdot [\mathbf{P}^{(1)}(f_1) \times \mathbf{P}^{(2)}(f_2, f_3)] \\ &+ \mathbf{Q}^{(2)}(f_2, f_3 + f_1) \cdot [\mathbf{P}^{(1)}(f_2) \times \mathbf{P}^{(2)}(f_3, f_1)] \cdot \mathbf{\Phi}_{231} \\ &+ \mathbf{Q}^{(2)}(f_1 + f_2, f_3) \cdot [\mathbf{P}^{(2)}(f_1, f_2) \times \mathbf{P}^{(1)}(f_3)]\} \\ &+ \mathbf{Q}^{(3)}(f_1, f_2, f_3) \cdot [\mathbf{P}^{(1)}(f_1) \times \mathbf{P}^{(1)}(f_2) \times \mathbf{P}^{(1)}(f_3)], \end{aligned} \tag{42c†}$$

where $\mathbf{\Phi}_{231}$ is defined in (68) and (69).

If the first system, $\{\mathbf{P}^{(k)}_{m,n}\}$, is linear, (42) reduces to

$$\mathbf{T}^{(k)}(f_1, \cdots, f_k) = \mathbf{Q}^{(k)}(f_1, \cdots, f_k)\cdot[\mathbf{P}^{(1)}(f_1) \times \cdots \times \mathbf{P}^{(1)}(f_k)]. \quad (43)$$

On the other hand, if the second system, $\{\mathbf{Q}^{(k)}_{l,m}\}$, is linear, (42) reduces to

$$\mathbf{T}^{(k)}(f_1, \cdots, f_k) = \mathbf{Q}^{(k)}(f_1 + \cdots + f_k)\cdot\mathbf{P}^{(k)}(f_1, \cdots, f_k). \quad (44)$$

6.4 Inversion

Let the numbers of inputs and outputs in the system represented by (35) be equal, i.e., $m = n$. Suppose that it is required to find the input vector, $\mathbf{U}_n$, in terms of the output vector, $\mathbf{W}_n$. This inversion operation is represented by

$$\mathbf{U}_n = \{\mathbf{P}^{(k)}_{n,n}\}^{-1} o \mathbf{W}_n = \{\mathbf{Q}^{(k)}_{n,n}\} o \mathbf{W}_n. \quad (45)$$

To find the kernels of the inverse system,

$$\{\mathbf{Q}^{(k)}_{n,n}\} = \{\mathbf{P}^{(k)}_{n,n}\}^{-1}, \quad (46)$$

it is helpful to use the interpretation given in Fig. 5, which defines the inversion operation in terms of the cascade operation and the identity system, $\{\mathbf{1}_n\}$, where $\mathbf{1}_n$ is the $n \times n$ identity matrix. Thus, applying the symmetric cascade relations of (42) to Fig. 5b by interchanging the roles of $\mathbf{P}$ and $\mathbf{Q}$, setting $\mathbf{T}^{(1)} = \mathbf{1}_n$, and $\mathbf{T}^{(k)} = \mathbf{0}$ for $k > 1$, and solving for $\mathbf{Q}^{(k)}$, one obtains the symmetric inversion relations

$$\mathbf{Q}^{(1)}(f_1) = [\mathbf{P}^{(1)}(f_1)]^{-1}, \quad (47a)$$

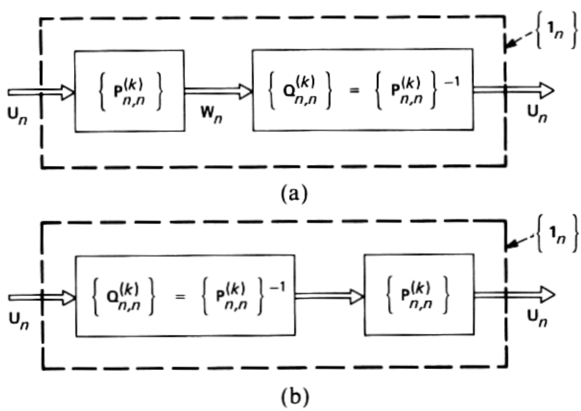

Fig. 5—Two equivalent interpretations of the inversion operation $\{\mathbf{Q}^{(k)}_{n,n}\} = \{\mathbf{P}^{(k)}_{n,n}\}^{-1}$: (a) $\{\mathbf{Q}^{(k)}_{n,n}\} * \{\mathbf{P}^{(k)}_{n,n}\} = \{\mathbf{1}_n\}$, and (b) $\{\mathbf{P}^{(k)}_{n,n}\} * \{\mathbf{Q}^{(k)}_{n,n}\} = \{\mathbf{1}_n\}$, where $\{\mathbf{1}_n\}$ is the identity system.

$$\mathbf{Q}^{(2)}(f_1, f_2) = -\mathbf{Q}^{(1)}(f_1 + f_2)\cdot\mathbf{P}^{(2)}(f_1, f_2)\cdot[\mathbf{Q}^{(1)}(f_1) \times \mathbf{Q}^{(1)}(f_2)], \quad (47b)$$

$$\begin{aligned}\mathbf{Q}^{(3)}(f_1, f_2, f_3) = -\mathbf{Q}^{(1)}(f_1 + f_2 + f_3)\cdot(⅔\{&\mathbf{P}^{(2)}(f_1, f_2 + f_3)\\ &\cdot[\mathbf{Q}^{(1)}(f_1) \times \mathbf{Q}^{(2)}(f_2, f_3)]\\ &+ \mathbf{P}^{(2)}(f_2, f_3 + f_1)\cdot[\mathbf{Q}^{(1)}(f_2) \times \mathbf{Q}^{(2)}(f_3, f_1)]\cdot\mathbf{\Phi}_{231}\\ &+ \mathbf{P}^{(2)}(f_1 + f_2, f_3)\cdot[\mathbf{Q}^{(2)}(f_1, f_2) \times \mathbf{Q}^{(1)}(f_3)]\}\\ &+ \mathbf{P}^{(3)}(f_1, f_2, f_3)\cdot[\mathbf{Q}^{(1)}(f_1) \times \mathbf{Q}^{(1)}(f_2) \times \mathbf{Q}^{(1)}(f_3)]). \quad (47c)\end{aligned}$$

Note that the inverse system exists if and only if $\mathbf{P}^{(1)}(f)$ is nonsingular.

6.5 Feedback

As an application of the three system operations discussed in the previous subsections, consider the nonlinear, feedback, MIMO system shown schematically in Fig. 6, where both the forward, $\{\mathbf{P}^{(k)}_{m,n}\}$, and reverse, $\{\mathbf{Q}^{(k)}_{n,m}\}$, branches are nonlinear. Using the operational notation of (35), one obtains

$$\mathbf{W}_m = \{\mathbf{P}^{(k)}_{m,n}\}o\mathbf{X}_n, \quad (48a)$$

$$\mathbf{X}_n = \mathbf{U}_n + \{\mathbf{Q}^{(k)}_{n,m}\}o\mathbf{W}_m, \quad (48b)$$

where $\mathbf{U}_n$, $\mathbf{X}_n$ and $\mathbf{W}_m$ are the $n \times 1$ input vector, the $n \times 1$ intermediate vector, and the $m \times 1$ output vector, respectively. Substituting $\mathbf{W}_m$ from (48a) into (48b), solving for $\mathbf{X}_n$ in terms of $\mathbf{U}_n$, and substituting the result in (48a), one obtains the feedback system equation

$$\mathbf{W}_m = \{\mathbf{F}^{(k)}_{m,n}\}o\mathbf{U}_n, \quad (49)$$

where

$$\{\mathbf{F}^{(k)}_{m,n}\} = \{\mathbf{P}^{(k)}_{m,n}\} * [\{\mathbf{1}_n\} - \{\mathbf{Q}^{(k)}_{n,m}\} * \{\mathbf{P}^{(k)}_{m,n}\}]^{-1}. \quad (50)$$

Thus, the kernels of the feedback system can be obtained by applying the subtraction, cascade, and inversion operations discussed above. However, the explicit formulas for these kernels will not be given here.

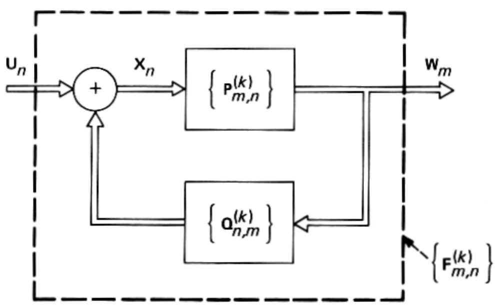

Fig. 6—A schematic representation of a nonlinear MIMO feedback system.

Actually, special cases of these formulas have been obtained for scalar systems in Refs. 2, 6, and 7.

Note from (50) and Section 6.4, that the feedback kernels exist if and only if the $n \times n$ matrix $[\mathbf{1}_n - \mathbf{Q}_{n,m}^{(1)}(f) \cdot \mathbf{P}_{m,n}^{(1)}(f)]$ is nonsingular. Note also that if $m = n$, and if $\mathbf{P}_{n,n}^{(1)}(f)$ is nonsingular, then (50) reduces to

$$\{\mathbf{F}_{n,n}^{(k)}\} = [\{\mathbf{P}_{n,n}^{(k)}\}^{-1} - \{\mathbf{Q}_{n,n}^{(k)}\}]^{-1}. \quad (51)$$

If the system in Fig. 6 is changed to a negative feedback system, then the minus signs in (50) and (51) should be changed to plus signs.

VII. CONCLUSIONS

A method of analysis has been presented for mildly nonlinear MIMO systems with memory. The method utilizes Volterra series whose kernels are two-dimensional matrices. The analysis was made possible through the use of the Kronecker product of matrices, which is a simple but powerful tool in matrix theory. This results in a compact representation of the system equations, and facilitates the systematic performance of various useful system operations, such as addition, cascading, inversion, and feedback. These operations can be used to describe a complex, nonlinear MIMO system as an interconnection of simple subsystems.

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APPENDIX A

Index Notation

Here we rewrite, in the index notation, some of the key equations marked by a dagger (†) in the body of the paper. The same equation numbers are used here as are used in the text. Before doing so, however, we note from (7) that, for MIMO systems with memory, the matrix kernels used in the matrix notations are related to the array kernels used in the index notation by the relations

$$[\mathbf{p}^{(k)}(\tau_1, \cdots, \tau_k)]_{ij} = p^{(k)}_{ij_1 \cdots j_k}(\tau_1, \cdots, \tau_k), \tag{52}$$

$$[\mathbf{P}^{(k)}(f_1, \cdots, f_k)]_{ij} = P^{(k)}_{ij_1 \cdots j_k}(f_1, \cdots, f_k), \tag{53}$$

where j is given by (6).

A list of the equations in question follows.

$$w_i^{(k)}(t_1, \cdots, t_k) = \sum_{j_1=1}^{m} \cdots \sum_{j_k=1}^{m} \left\{ \int_{-\infty}^{\infty} \cdots \int p_{ij_1 \cdots j_k}^{(k)}(\tau_1, \cdots, \tau_k) \cdot u_{j_1}(t_1 - \tau_1) \cdots u_{j_k}(t_k - \tau_k) d\tau_1 \cdots d\tau_k \right\} \quad (12)^\dagger$$

$$W_i^{(k)}(f_1, \cdots, f_k) = \sum_{j_1=1}^{m} \cdots \sum_{j_k=1}^{m} P_{ij_1 \cdots j_k}^{(k)}(f_1, \cdots, f_k) \cdot U_{j_1}(f_1) \cdots U_{j_k}(f_k). \quad (16)^\dagger$$

$$P_{ij_1 j_2}^{(2)}(f_1, f_2) = P_{ij_2 j_1}^{(2)}(f_2, f_1). \quad (21a)^\dagger$$

$$P_{ij_1 j_2 j_3}^{(3)}(f_1, f_2, f_3) = P_{ij_\alpha j_\beta j_\gamma}^{(3)}(f_\alpha, f_\beta, f_\gamma). \quad (21b)^\dagger$$

$$P_{ij_1 j_2}^{(2)}(f_1, f_2) = \tfrac{1}{2}[\hat{P}_{ij_1 j_2}^{(2)}(f_1, f_2) + \hat{P}_{ij_2 j_1}^{(2)}(f_2, f_1)]. \quad (22a)^\dagger$$

$$P_{ij_1 j_2 j_3}^{(3)}(f_1, f_2, f_3) = \frac{1}{6} \sum_{\alpha,\beta,\gamma} \hat{P}_{ij_\alpha j_\beta j_\gamma}^{(3)}(f_\alpha, f_\beta, f_\gamma). \quad (22b)^\dagger$$

$$P_{ij_1 j_2}^{(2)}(f, f) = P_{ij_2 j_1}^{(2)}(f, f). \quad (30a)^\dagger$$

$$P_{ij_1 j_2}^{(2)}(f, -f) = [P_{ij_2 j_1}^{(2)}(f, -f)]^*. \quad (30b)^\dagger$$

$$P_{ij_1 j_2 j_3}^{(3)}(f, f, -f) = P_{ij_2 j_1 j_3}^{(3)}(f, f, -f). \quad (30c)^\dagger$$

$$P_{ij_1 j_2 j_3}^{(3)}(f, f, f) = P_{ij_\alpha j_\beta j_\gamma}^{(3)}(f, f, f). \quad (30d)^\dagger$$

$$T_{ij_1}^{(1)}(f_1) = \sum_{\alpha=1}^{m} Q_{i\alpha}^{(1)}(f_1) P_{\alpha j_1}^{(1)}(f_1). \quad (42a)^\dagger$$

$$T_{ij_1 j_2}^{(2)}(f_1, f_2) = \sum_{\alpha=1}^{m} Q_{i\alpha}^{(1)}(f_1 + f_2) P_{\alpha j_1 j_2}^{(2)}(f_1, f_2) + \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} Q_{i\alpha\beta}^{(2)}(f_1, f_2) P_{\alpha j_1}^{(1)}(f_1) P_{\beta j_2}^{(1)}(f_2). \quad (42b)^\dagger$$

$$\hat{T}_{ij_1 j_2 j_3}^{(3)}(f_1, f_2, f_3) = \sum_{\alpha=1}^{m} Q_{i\alpha}^{(1)}(f_1 + f_2 + f_3) P_{\alpha j_1 j_2 j_3}^{(3)}(f_1, f_2, f_3) + \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} [Q_{i\alpha\beta}^{(2)}(f_1, f_2 + f_3) P_{\alpha j_1}^{(1)}(f_1) P_{\beta j_2 j_3}^{(2)}(f_2, f_3) + Q_{i\alpha\beta}^{(2)}(f_1 + f_2, f_3) P_{\alpha j_1 j_2}^{(2)}(f_1, f_2) P_{\beta j_3}^{(1)}(f_3)] + \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} \sum_{\gamma=1}^{m} Q_{i\alpha\beta\gamma}^{(3)}(f_1, f_2, f_3) P_{\alpha j_1}^{(1)}(f_1) P_{\beta j_2}^{(1)}(f_2) P_{\gamma j_3}^{(1)}(f_3). \quad (42\hat{c})^\dagger$$

$$T^{(3)}_{ij_1j_2j_3}(f_1, f_2, f_3) = \sum_{\alpha=1}^{m} Q^{(1)}_{i\alpha}(f_1 + f_2 + f_3)P^{(3)}_{\alpha j_1 j_2 j_3}(f_1, f_2, f_3)$$

$$+ \frac{2}{3}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{m} [Q^{(2)}_{i\alpha\beta}(f_1, f_2 + f_3)P^{(1)}_{\alpha j_1}(f_1)P^{(2)}_{\beta j_2 j_3}(f_2, f_3)$$

$$+ Q^{(2)}_{i\alpha\beta}(f_2, f_3 + f_1)P^{(1)}_{\alpha j_2}(f_2)P^{(2)}_{\beta j_3 j_1}(f_3, f_1)$$

$$+ Q^{(2)}_{i\alpha\beta}(f_3, f_1 + f_2)P^{(1)}_{\alpha j_3}(f_3)P^{(2)}_{\beta j_1 j_2}(f_1, f_2)]$$

$$+ \sum_{\alpha=1}^{m}\sum_{\beta=1}^{m}\sum_{\gamma=1}^{m} Q^{(3)}_{i\alpha\beta\gamma}(f_1, f_2, f_3)P^{(1)}_{\alpha j_1}(f_1)P^{(1)}_{\beta j_2}(f_2)P^{(1)}_{\gamma j_3}(f_3). \quad (42c)^\dagger$$

APPENDIX B

Kronecker Product of Matrices

Here we define the Kronecker product of matrices and summarize some of its properties that are used in this paper. More extensive coverage of this topic is given in Refs. 22 through 24.

Let $\mathbf{A} = [\alpha_{i_a j_a}]$ and $\mathbf{B} = [b_{i_b j_b}]$ be $m_a \times n_a$ and $m_b \times n_b$ matrices, respectively. Their Kronecker product results in the $m_a m_b \times n_a n_b$ matrix, $\mathbf{C} = [c_{i_c j_c}]$, given by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{bmatrix} \mathbf{A}b_{11} & \mathbf{A}b_{12} & \cdots & \mathbf{A}b_{1n_b} \\ \mathbf{A}b_{21} & \mathbf{A}b_{22} & & \mathbf{A}b_{2n_b} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}b_{m_b 1} & \mathbf{A}b_{m_b 2} & \cdots & \mathbf{A}b_{m_b n_b} \end{bmatrix}, \quad (54)$$

where "×" is the Kronecker-product symbol. Thus,

$$c_{i_c j_c} = a_{i_a j_a} b_{i_b j_b}, \quad (55a)$$

where

$$i_c = i_a + m_a(i_b - 1), \quad (55b)$$

$$j_c = j_a + n_a(j_b - 1). \quad (55c)$$

Note that, since $i_a \le m_a$ and $j_a \le n_a$, (55b) and (55c) have unique solutions for i_a, i_b, j_a and j_b in terms of i_c and j_c. Actually, (54) and (55) define the left Kronecker product.[22] One can also define a right Kronecker product,[23,24] which, however, is not used in this paper. In general,

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}. \quad (56)$$

It can be shown that the Kronecker product has the following properties:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} \equiv \mathbf{A} \times \mathbf{B} \times \mathbf{C}. \tag{57}$$

$$(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \times \mathbf{C}) + (\mathbf{B} \times \mathbf{C}). \tag{58}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}). \tag{59}$$

$$(\mathbf{A} \cdot \mathbf{B}) \times (\mathbf{C} \cdot \mathbf{D}) = (\mathbf{A} \times \mathbf{C}) \cdot (\mathbf{B} \times \mathbf{D}). \tag{60}$$

$$(\mathbf{A} \times \mathbf{B})^{-1} = \mathbf{A}^{-1} \times \mathbf{B}^{-1}. \tag{61}$$

$$(\mathbf{A} \times \mathbf{B})^{T} = \mathbf{A}^{T} \times \mathbf{B}^{T}. \tag{62}$$

In the above equations, "T" refers to matrix transposition, and the dot implies ordinary matrix multiplication. The dimensions of the various matrices are arbitrary, but of course, should be consistent with the requirements of the inversion, addition, and ordinary multiplication operations, where applicable.

APPENDIX C

Reversing and Permutation Matrices

Here we define the $n^2 \times n^2$ reversing matrix, $\mathbf{R}^{(n)}$, and the six $n^3 \times n^3$ permutation matrices, $\mathbf{\Phi}^{(n)}_{\alpha\beta\gamma}$, which satisfy (19) and (20). The superscript "(n)" is used in this appendix to emphasize the dimensions. It can be shown from (19) and (55) that $\mathbf{R}^{(n)}$ is given by (cf. Ref. 24)

$$R^{(n)}_{i+n(j-1),\,k+n(l-1)} = \delta_{il}\delta_{jk}, \qquad i, j, k, l = 1, 2, \cdots, n, \tag{63}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, which is equal to 1 if $\alpha = \beta$, and 0 if $\alpha \neq \beta$. For example,

$$\mathbf{R}^{(2)} = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix}. \tag{64}$$

It can be verified that

$$\mathbf{R}^{(n)} = [\mathbf{R}^{(n)}]^{T} = [\mathbf{R}^{(n)}]^{-1}, \tag{65}$$

where "T" refers to matrix transposition. Moreover, if $\mathbf{M}_1$ and $\mathbf{M}_2$ are $m \times n$ matrices, then

$$\mathbf{R}^{(m)} \cdot (\mathbf{M}_1 \times \mathbf{M}_2) \cdot \mathbf{R}^{(n)} = \mathbf{M}_2 \times \mathbf{M}_1, \tag{66}$$

which is a generalization of (19).

It can be shown from (19), (20) and (60) that

$$\mathbf{\Phi}^{(n)}_{123} = [\mathbf{\Phi}^{(n)}_{123}]^{T} = [\mathbf{\Phi}^{(n)}_{123}]^{-1} = \mathbf{1}_{n^3}, \tag{67a}$$

$$\mathbf{\Phi}^{(n)}_{132} = [\mathbf{\Phi}^{(n)}_{132}]^{T} = [\mathbf{\Phi}^{(n)}_{132}]^{-1} = \mathbf{1}_{n} \times \mathbf{R}^{(n)}, \tag{67b}$$

$$\mathbf{\Phi}^{(n)}_{213} = [\mathbf{\Phi}^{(n)}_{213}]^{T} = [\mathbf{\Phi}^{(n)}_{213}]^{-1} = \mathbf{R}^{(n)} \times \mathbf{1}_{n}, \tag{67c}$$

$$\mathbf{\Phi}_{231}^{(n)} = [\mathbf{\Phi}_{312}^{(n)}]^T = [\mathbf{\Phi}_{312}^{(n)}]^{-1} = [\mathbf{1}_n \times \mathbf{R}^{(n)}]\cdot[\mathbf{R}^{(n)} \times \mathbf{1}_n], \tag{67d}$$

$$\mathbf{\Phi}_{312}^{(n)} = [\mathbf{\Phi}_{231}^{(n)}]^T = [\mathbf{\Phi}_{231}^{(n)}]^{-1} = [\mathbf{R}^{(n)} \times \mathbf{1}_n]\cdot[\mathbf{1}_n \times \mathbf{R}^{(n)}], \tag{67e}$$

$$\begin{aligned}\mathbf{\Phi}_{321}^{(n)} &= [\mathbf{\Phi}_{321}^{(n)}]^T = [\mathbf{\Phi}_{321}^{(n)}]^{-1} = [\mathbf{1}_n \times \mathbf{R}^{(n)}]\cdot[\mathbf{R}^{(n)} \times \mathbf{1}_n]\cdot[\mathbf{1}_n \times \mathbf{R}^{(n)}] \\ &= [\mathbf{R}^{(n)} \times \mathbf{1}_n]\cdot[\mathbf{1}_n \times \mathbf{R}^{(n)}]\cdot[\mathbf{R}^{(n)} \times \mathbf{1}_n],\end{aligned} \tag{67f}$$

where $\mathbf{1}_n$ and $\mathbf{1}_{n^3}$ are the $n \times n$ and $n^3 \times n^3$ identity matrices, respectively. Also, it can be shown from (20) and (55) that, if α, β, γ are any permutation of 1, 2, 3, then the i-j element of $\mathbf{\Phi}_{\alpha\beta\gamma}^{(n)}$ is given by

$$[\mathbf{\Phi}_{\alpha\beta\gamma}^{(n)}]_{ij} = \delta_{i_1 j_1}\delta_{i_2 j_2}\delta_{i_3 j_3}, \tag{68a}$$

where

$$i = i_\alpha + n(i_\beta - 1) + n^2(i_\gamma - 1), \tag{68b}$$

$$j = j_1 + n(j_2 - 1) + n^2(j_3 - 1), \tag{68c}$$

and where $i_1, j_1, i_2, j_2, i_3, j_3 = 1, 2, \cdots, n$. For example, (67d) and (68) gives

$$\mathbf{\Phi}_{231}^{(2)} = [\mathbf{\Phi}_{312}^{(2)}]^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{69}$$

It can be verified that if $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ are $m \times n$ matrices, then

$$\mathbf{\Phi}_{\alpha\beta\gamma}^{(m)}\cdot(\mathbf{M}_1 \times \mathbf{M}_2 \times \mathbf{M}_3)\cdot[\mathbf{\Phi}_{\alpha\beta\gamma}^{(n)}]^T = \mathbf{M}_\alpha \times \mathbf{M}_\beta \times \mathbf{M}_\gamma, \tag{70}$$

which is a generalization of (55). Also, if $\mathbf{M}$ and $\mathbf{N}$ are $m \times n$ and $m^2 \times n^2$ matrices, respectively, then

$$\mathbf{\Phi}_{231}^{(m)}\cdot(\mathbf{M} \times \mathbf{N})\cdot[\mathbf{\Phi}_{231}^{(n)}]^T = \mathbf{N} \times \mathbf{M}, \tag{71a}$$

$$\mathbf{\Phi}_{312}^{(m)}\cdot(\mathbf{N} \times \mathbf{M})\cdot[\mathbf{\Phi}_{312}^{(n)}]^T = \mathbf{M} \times \mathbf{N}. \tag{71b}$$

Moreover, if $\mathbf{M}$ and $\mathbf{K}$ are $m \times n$ and $m \times n^2$ matrices, respectively, then

$$\mathbf{R}^{(m)}\cdot(\mathbf{M} \times \mathbf{K})\cdot[\mathbf{\Phi}_{231}^{(n)}]^T = \mathbf{K} \times \mathbf{M}, \tag{72a}$$

$$\mathbf{R}^{(m)}\cdot(\mathbf{K} \times \mathbf{M})\cdot[\mathbf{\Phi}_{312}^{(n)}]^T = \mathbf{M} \times \mathbf{K}. \tag{72b}$$

Finally, if $\mathbf{M}$ and $\mathbf{L}$ are $m \times n$ and $m^2 \times n$ matrices, respectively, then

$$\mathbf{\Phi}_{231}^{(m)}\cdot(\mathbf{M} \times \mathbf{L})\cdot\mathbf{R}^{(n)} = \mathbf{L} \times \mathbf{M}, \tag{73a}$$

$$\mathbf{\Phi}_{312}^{(m)}\cdot(\mathbf{L} \times \mathbf{M})\cdot\mathbf{R}^{(n)} = \mathbf{M} \times \mathbf{L}. \tag{73b}$$

APPENDIX D

General Cascade Relation

Here we give a generalization of the unsymmetric cascade relations given in (42a), (42b), and (42$\hat{c}$) for arbitrary k, (cf. Refs. 7 and 9 for scalar systems)

$$\hat{\mathbf{T}}^{(k)}(f_1, \cdots, f_k) = \sum_{l=1}^{k} \left\{ \sum_{\substack{k_1, k_2, \cdots, k_l = 1 \\ (k_1 + k_2 + \cdots + k_l = k)}}^{k-l+1} \mathbf{Q}^{(l)}(f_1 + \cdots + f_{k_1}, f_{k_1+1} + \cdots + f_{k_1+k_2}, \cdots, f_{k-k_l+1} + \cdots + f_k)[\mathbf{P}^{(k_1)}(f_1, \cdots, f_{k_1}) \times \mathbf{P}^{(k_2)}(f_{k_1+1}, \cdots, f_{k_1+k_2}) \times \cdots \times \mathbf{P}^{(k_l)}(f_{k-k_l+1}, \cdots, f_k)] \right\}. \quad (74)$$

Note that the second summation contains $\binom{k-1}{l-1}$ terms, and that the frequency arguments always appear in the order $f_1, f_2, \cdots, f_k$. As is the case with (42$\hat{c}$), the cascade relation of (74) does not preserve kernel symmetry for $k \geq 3$. The symmetric form of (74), which would generalize (42c), will not be given since it requires the use of permutation matrices of more than three indices, which have not been introduced yet.