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Computing the Distribution of a Random Variable via Gaussian Quadrature Rules

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Using the technique of Gaussian quadrature rules, a new estimator is proposed for approximating the distribution of a random variable given only a finite number of its moments. The estimator is shown by numerous examples to be accurate on the tails of both continuous and discrete distributions. Efficient algorithms exist for computing the estimator from the first 2N moments of the random variable. A robust implementation of the estimator is presented, along with rules that provide additional protection against computer roundoff errors.

I. INTRODUCTION

In this paper we present a method for computing the Cumulative Distribution Function (CDF) of an arbitrary random variable. Using the theory of Gaussian Quadrature Rules (GQRS), we derive an estimator that converges asymptotically to the true CDF. In practice, convergence is obtained without excessive computation. A general estimator is developed here that is applicable to a wide class of problems.

Section 2.1 begins with a review of GQR analysis as it has traditionally been used for numerical integration. Several authors have shown the existence of extremely efficient algorithms for computing the parameters of the GQR. An efficient and robust procedure for obtaining the GQR parameters is presented in the appendix. Two CDF estimators based on GQR are derived in Sections 2.3 and 2.4. The first estimator is most suited to numerical integration schemes and estimation of discrete distributions, while the second is appropriate for continuous distributions such as Gaussian noise or crosstalk. Section III gives numerous examples that show the inherent accuracy of the technique for continuous, discrete, and mixed distributions. Computational methods for deriving the required moments are discussed, along with modifications that tend to mitigate the roundoff errors that plague GQR analysis of nonsymmetric distributions.

II. THEORY AND PROPERTIES OF GQR

2.1 Classical use of GQR

The GQR has traditionally been used as a numerical integration procedure and is particularly efficient for computing integrals of the form

$$\int_a^b f(x)w(x)dx$$

where the integrand has been factored into a non-negative term $w(x)$ and a strongly continuous term $f(x)$.

The first application of GQR in the communications literature[1] was motivated by the work of Golub and Welsch[2] and Sack and Donovan[3], who showed that the non-negative factor $w(x)$ need not even be completely known to compute the desired integral. Only a finite number of the moments of $w(x)$ are required to find the desired integration rule. Benedetto et al.[1] noticed that the problem of error probability evaluation in the presence of intersymbol interference (ISI) could be posed in this form. Other applications of the GQR technique can be found in Refs. 4 through 9.

In this paper, we apply the GQR technique to a larger class of problems where $f(x)$ need not be continuous. We begin by reviewing a fundamental result in the theory of GQR.

Theorem: *Let $w(x)$ be a non-negative weight function defined on (a, b). Then if $f(x)$ has continuous derivatives up to order $2N$ (see Refs.* 10 *through* 13),

$$\begin{aligned} J &= \int_a^b f(x)w(x)dx \\ &= \sum_{i=1}^{N} A_i f(t_i) + R_N(\xi) \qquad a < \xi < b \\ & \alpha < t_i < b \qquad i = 1, 2 \cdots N, \end{aligned} \tag{1}$$

where

$$R_N(\xi) = \frac{f^{(2N)}(\xi)}{(2N)!(k_N)^2}, \qquad a < \xi < b, \tag{2}$$

$f^{(2N)}(x)$ is the $2N$th *derivative of $f(x)$ and $(2N)!$ is $2N$ factorial. The nodes $\{t_i\}$ are the distinct real roots of the unique* Nth *degree polynomial*

$$p_N(x) = k_N \prod_{i=1}^{N} (x - t_i), \qquad k_N > 0. \tag{3}$$

The polynomials $p_n(x)$ are orthonormal with respect to $w(x)$, i.e.,

$$\int_a^b w(x)p_m(x)p_n(x)dx = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases}$$

The strictly positive weights (or Christoffel numbers) are in turn given by

$$A_i = \frac{-k_{N+1}}{k_N} \frac{1}{p_{N+1}(t_i)p'_N(t_i)} \qquad i = 1, 2 \cdots N, \tag{4}$$

where

$$p'_N(t_i) = \left.\frac{dp_N(t)}{dt}\right|_{t=t_i}.$$

The $2N$-tuple $\{A_i, t_i\}_{i=1}^{N}$ is known as the N-point rule corresponding to $w(x)$.

If $f(x)$ is a polynomial of degree $(2N - 1)$ or less, the remainder $R_N(\xi)$ equals zero and the GQR is exact. This affords the maximum degree of precision (i.e., the maximum degree polynomial that is integrable with no error for an N-point rule) possible with a quadrature formula of the form of (1).[10–12] When the remainder is not zero, it can be bounded in magnitude to obtain upper and lower bounds on J. The bounds obtained in Ref. 1 for the ISI and Gaussian noise problem are often loose though, and convergence of the N-term summation in (2) is usually much faster than might be inferred from bounds on $R_N(\xi)$.

2.2 Methods for computing GQR

Several algorithms are known for efficiently computing the rule for an arbitrary weight function $w(x)$. Extremely useful procedures have been discovered by Golub and Welsh,[2] Sack and Donovan,[3] and Gautchi.[14] The outstanding merit of these techniques is that the N-point rule corresponding to a given $w(x)$ can be computed from the moments

$$\mu_i = \int_a^b x^i w(x)dx. \tag{5}$$

Because explicit knowledge of the weight function $w(x)$ is not required, the GQR procedure is a powerful tool for the analysis of communications systems.

Details of an algorithm for computing GQR are given in the appendix. Our algorithm is a modification of Gautchi's procedure,[14] which tends to reduce computer roundoff errors. The critical stage in the algorithm is the Cholesky decomposition of a positive definite matrix of moments. The standard Cholesky decomposition used in Refs. 1 and 2 fails when, because of limitations of machine accuracy, the matrix is no longer positive definite due to roundoff errors. Improved accuracy is obtained by using an alternate method of performing the Cholesky decomposition* that avoids taking a square root at each step in the algorithm.[15] Combining the alternate Cholesky decomposition with the modified moment algorithm of Gautchi[14] yields an extremely stable method for obtaining GQR. Further discussion of techniques to mitigate computer roundoff errors is found in the appendix.

2.3 Computing the distribution of a random variable via GQR

In Ref. 1 GQRs are used to obtain the exact probability of error for digital transmission in the presence of ISI and Gaussian noise. The problem was reduced, via the GQR approach, to computing the moments of the ISI and letting $f(x)$ in (1) be the probability of error caused by Gaussian noise conditioned on the ISI. The ISI moments can be computed via Prabhu's method[16] when the data symbols are independent. For a large class of correlated data, the moments can be efficiently computed via the modified Cariolaro-Pupolin algorithm.[17,18] Both of the above procedures are easily implemented and have a complexity that grows only linearly with pulse duration.

While there have been numerous applications of GQR to problems in the literature, all those known to us have had the restriction that the function $f(x)$ has continuous derivatives up to order $2N$. Presumably, this is because of the desire for strict bounds on the error term in (2). If we are willing to forego the analytical error term and consequently accept an empirical convergence of (1), we can apply the GQR technique to a larger class of problems with excellent results.

The following theorem shows that no continuity requirements need be imposed on $f(x)$.

Theorem: (see Ref. 19) *If $W(x)$ is a fixed, nondecreasing function with infinitely many points of increase and the Riemann-Stieltjes integral*

$$\int_a^b f(x)dW(x)$$

exists, then

$$\int_a^b f(x)dW(x) = \lim_{N->\infty} \sum_{i=1}^{N} A_i f(t_i), \tag{6}$$

* Applying the alternate Cholesky decomposition to the GQR problem was suggested by L. Kaufman. Subsequently, the same approach was found to have been independently proposed in Ref. 4.

where $\{A_i, t_i\}_{i=1}^{N}$ *is the* GQR *corresponding to the moments*

$$\mu_i = \int_a^b x^i dW(x) \qquad i = 0, 1, 2, \cdots 2N.$$

The function $f(x)$ is arbitrary as long as the integral in (6) exists.

Because the CDF of a random variable is a nondecreasing function, we write the statistical expectation of the function $f(x)$ as

$$E[f(x)] = \int_a^b f(x)dW(x), \tag{7}$$

where $W(x)$ is a probability measure with infinitely many points of rise. Choosing $f(x)$ to be the indicator function

$$\begin{aligned} f(x) &= \phi_\alpha(x) \\ &= \begin{cases} 1 & x \leq \alpha \\ 0 & x > \alpha, \end{cases} \end{aligned} \tag{8}$$

we obtain the distribution function of the random variable as

$$\begin{aligned} \int_a^b \phi_\alpha(x)dW(x) &= \lim_{N->\infty} \sum_{S_N^\alpha} A_i \\ &= \lim_{N->\infty} \hat{W}_N(\alpha), \end{aligned} \tag{9}$$

where

$$\hat{W}_N(\alpha) = \sum_{S_N^\alpha} A_i \tag{10}$$

and

$$S_N^\alpha = \{i \mid t_i \leq \alpha\}$$

is the set of indices for which $t_i \leq \alpha$.

Since the rule can be obtained from the $\{\mu_i\}_{i=0}^{2N}$, we have a means of constructing an approximation to the CDF of a random variable from its moments. In the limit as N approaches infinity, eq. (9) is exact at each point α.

This leads us to propose the following estimator

$$W(x) \approx \hat{W}_N(x) = \sum_{S_N^x} A_i. \tag{11}$$

This estimator gives a staircase approximation to the true cumulative distribution that becomes increasingly fine as N increases. Equivalently, each (A_i, t_i) can be considered a point mass of a discrete approximation to the true probability density function.

While Szegö's theorem proves the asymptotic convergence of the estimator to the true CDF when $W(x)$ has an infinite number of points of rise, a different result holds for a discrete distribution with a finite number of points of increase.

Theorem: If $W(x)$ is a fixed, nondecreasing function with $M < \infty$ points of increase, then

$$\int_a^b \phi_a(x)dW(x) = \lim_{N\to M} \hat{W}_N(\alpha). \tag{12}$$

Proof: An alternate formulation of Gaussian Quadrature[14] is as the purely algebraic solution to

$$\mu_j = \sum_{i=1}^{N} A_i(t_i)^j \quad j = 0, 1, \cdot 2M. \tag{13}$$

Now we assume the unknown discrete PDF is of the form

$$w(x) = \sum_{i=1}^{M} A_i\delta(x - t_i).$$

The moments of this random variable are given by

$$\sum_{i=1}^{M} A_i(t_i)^j,$$

which is identical to (13) for $N = M$.

Finally, we consider the behavior of the GQR for N larger than the number of points of increase M. The result is that the algorithm breaks down entirely. This is because a discrete distribution that takes on exactly M values is completely characterized by its first $2M$ moments and the addition of redundant moments to the problem causes the procedure to fail when the Hankel matrix of moments [eq. (21)] becomes nonsingular.

2.4 A modified GQR estimator

The following is a modification of the estimator $\hat{W}_N(\alpha)$ that has been found to be more accurate in many applications. Instead of assuming that the approximation PDF is composed of point masses, we assume that each area of mass A_i is more accurately modeled by a narrow, even symmetric, distribution centered around the point t_i. Thus, we propose the smoothed estimator $W_N^*(\alpha)$ which, when evaluated at a node, equals

$$W_N^*(t_i) = \hat{W}_N(t_i) - \frac{A_i}{2}. \tag{14}$$

Between nodes, $W_N^*(\alpha)$ is given by any "smooth" interpolation routine. A simple linear interpolation was found to be sufficient in the examples

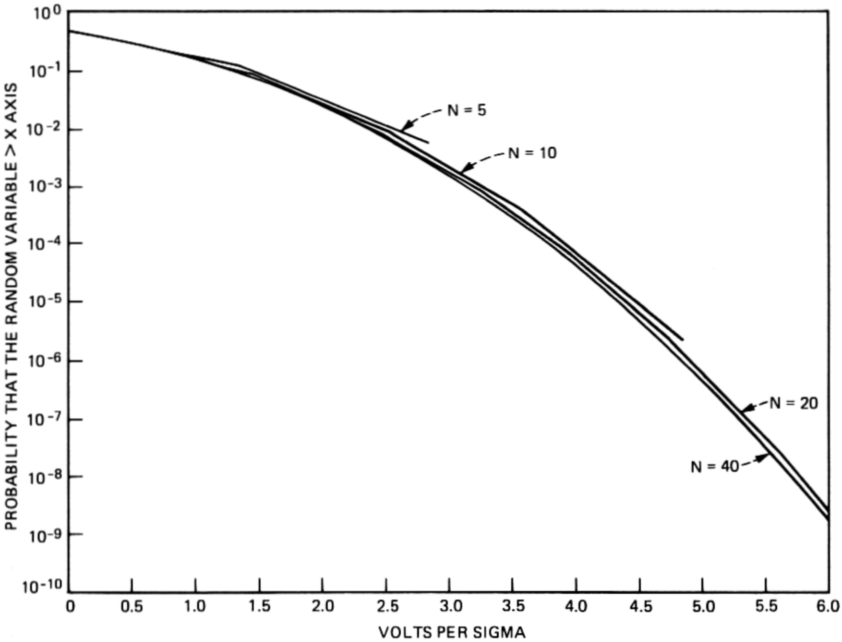

Fig. 1—Convergence of GQR estimator for Gaussian PDF.

that follow. This estimator does not have the jump discontinuities of the estimator $\hat{W}_N(\alpha)$ and is intuitively more satisfying because it fits a smoother distribution to $W(x)$. $W_N^*(\alpha)$ has been found to give more accurate results when applied to known continuous distributions and to discrete distributions when M is much greater than N.

III. APPLICATION TO ARBITRARY DISTRIBUTIONS

3.1 Known continuous random variable case

To show the convergence properties of the GQR technique, we illustrate the behavior of $\hat{W}_N(\alpha)$ and $W_N^*(\alpha)$ with some examples. We begin with the Gaussian distribution. Assuming a zero mean, unit variance random variable X, we compute the GQR estimators for various values of N in Fig. 1. Reasonably accurate results were obtained at the 10^{-6} point for $N > 10$. This empirical rate of convergence is also typical of distributions that have near-Gaussian statistics. The GQR algorithm, using the Cholesky decomposition described in the appendix, returned accurate results for all $N \leq 60$, where $N = 60$ was the dimensionality limit in the computer program.

In general, the GQR algorithm performs well for zero mean, symmetric distributions. To illustrate the problems that can occur with nonsymmetric distributions, consider the lognormal distribution related to the Gaussian distribution by $Y = e^X$. Straightforward computation of the moments yields

$$\mu_k = \exp(k^2/2). \tag{15}$$

Using these moments in the GQR algorithm, the algorithm breaks down at $N = 17$ because of roundoff errors in computation. This problem is solved by a transformation that symmetrizes the distribution. We then compute the GQR corresponding to the symmetrized distribution and take the inverse transform to obtain the original distribution.

For the lognormal distribution, we form a new PDF

$$w_e(y) = 1/2[w(y) + w(-y)], \tag{16}$$

which corresponds to the even part of $w(y)$. The moments of $w_e(y)$ are obtained by setting the odd moments of $w(y)$ equal to zero. The symmetric moments are then used in the GQR algorithm to obtain $W_e(y)$, which is easily transformed back to the desired CDF via the relation

$$W(y) = \begin{cases} 2W_e(y) - 1 & x \geq 0 \\ 0 & x < 0. \end{cases} \tag{17}$$

Experience with these procedures suggests that it is well worth the effort to transform distributions that are not symmetric and even (see Fig. 2). The modified moments produce the same robust accuracy seen with the Gaussian distribution above.

As another example, consider a uniform distribution defined on the interval $(-1, 1)$. The convergence of the GQR estimator $\hat{W}_N(\alpha)$ is shown in Fig. 3. Since the distribution has only finite support, by eq. (1) we know that all the nodes will lie in the interval $(-1, 1)$. In the limit as $N \to \infty$, the nodes will become more densely packed in this interval and

$$\lim_{i\to\infty} \{\max_i |t_i|\} = 1. \tag{18}$$

Thus, the GQR algorithm can be used to find the maximum value that a random variable attains, i.e., the largest node $t_{\max}$. This can be used, for example, to find the maximum eye degradation in a digital regenerator caused by correlated intersymbol interference.

All the examples so far have been trivial applications since we knew the real distributions *a priori*. A more interesting application is determining the distribution of the sum of K lognormal random variables. This problem has a long history and no closed form solution is known. This PDF is related to the distribution of crosstalk power in paired cable transmission systems and also results from transmission over certain types of fading channels. Utilizing the GQR technique, we can find the desired distribution if we can compute the necessary moments.

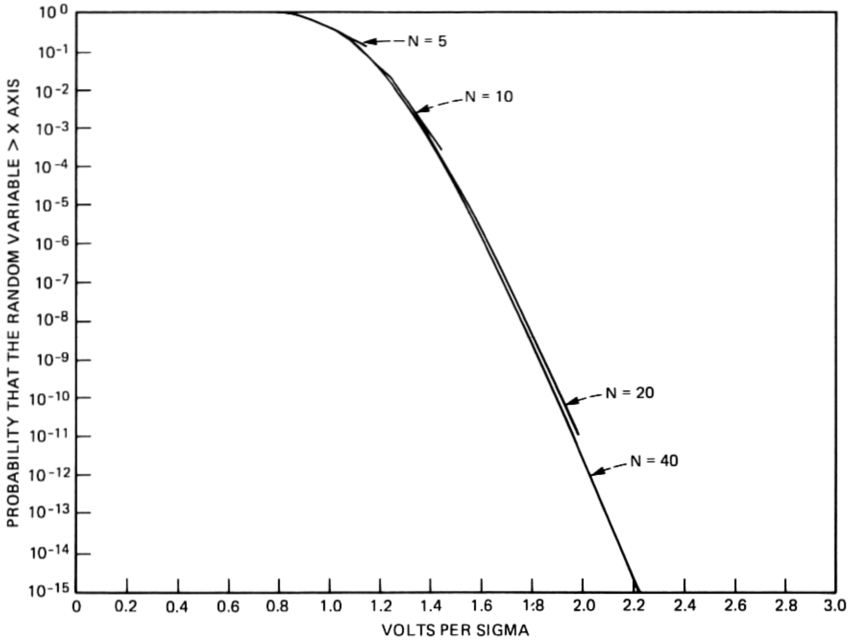

Fig. 2—Convergence of GQR estimator for lognormal PDF.

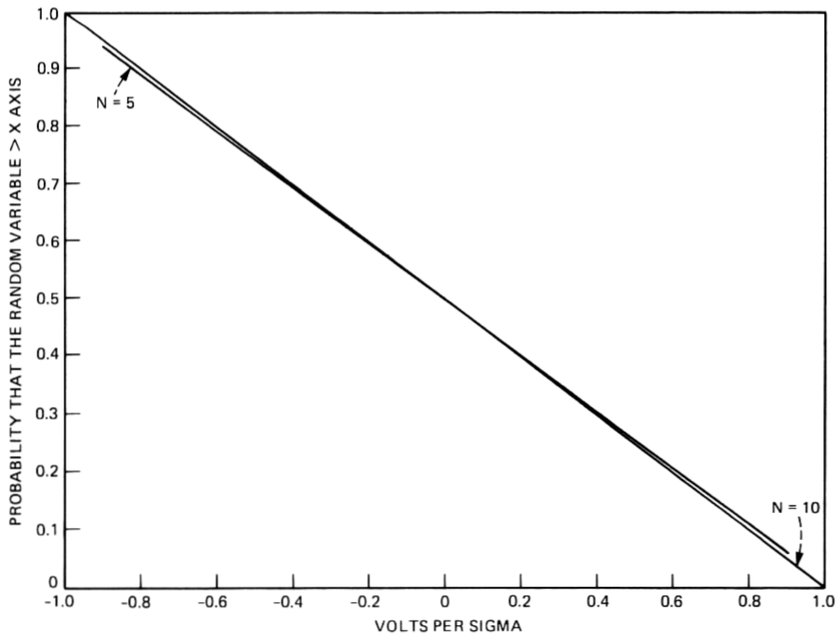

Fig. 3—Convergence of GQR estimator for uniform PDF.

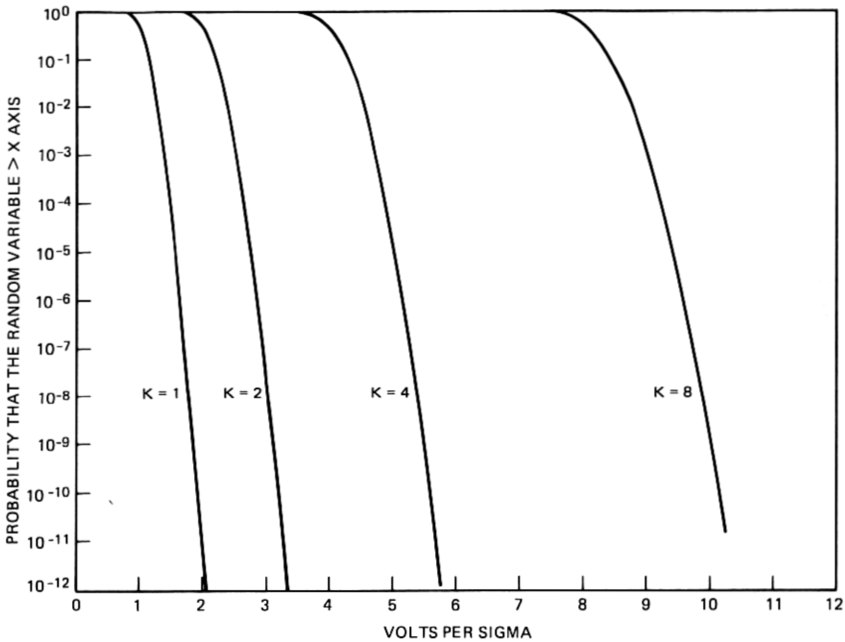

Fig. 4—Sum of K lognormal (K = 1, 2, 4, 8).

Assuming that the lognormal random variables are independent, and following Prabhu,[16] we find that the moments of

$$V_K = \sum_{i=1}^{K} Y_i$$

are given by the recurrence relation

$$E[(V_k)^i] = \sum_{l=0}^{i} \binom{i}{l} E[(V_{k-1})^l]\mu_{i-l}, \tag{19}$$

where $\{\mu_i\}_{i=1}^{2N}$ are the moments of the independent, identically distributed lognormal random variables. Figure 4 shows the resulting distributions for $K = 2, 4, 8$, and 16. As we mentioned above, the distribution was symmetrized and inverse transformed to reduce the effects of roundoff errors. This technique can be applied to any number of arbitrary distributions for which the required moments can be computed.

3.2 Known discrete random variable case

In this section, we apply the GQR estimator $W_N^*(\alpha)$ to discrete distributions. First we consider the case of a mixed distribution composed of a Gaussian distribution plus discrete components. The weights

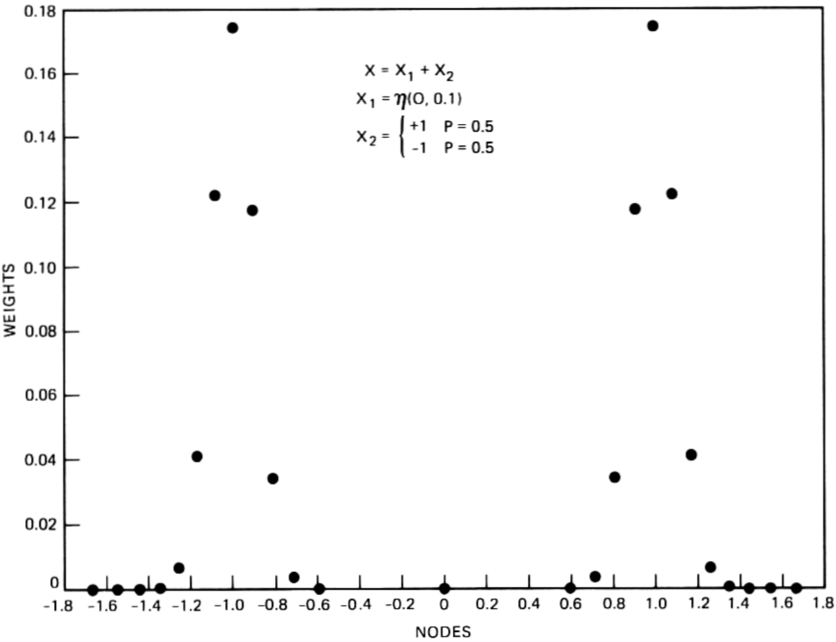

Fig. 5—Mixed Gaussian distribution.

and nodes of the estimator are shown in Fig. 5, where the second moment of the discrete part equals ten times that of the Gaussian component. As we can readily discern, the GQR procedure is useful in identifying the discrete components of a PDF.

As the final example of a known distribution, we consider the sum of nine equally spaced delta functions

$$w(x) = 1/9 \sum_{i=1}^{9} \delta(x - x_i) \qquad x_i = -5 + i \qquad i = 1, 2, \cdots 9. \tag{20}$$

The convergence of the GQR estimator $W_N^*(x)$ is shown in Fig. 6, where the $N = 9$ estimator is exact since the distribution is uniquely defined by the first $2N = 18$ moments. For $N > 9$, the algorithm breaks down.

IV. SUMMARY

An estimator based on GQR has been proposed, which converges rapidly to the CDF of a random variable and requires only knowledge of the moments of the random variable in question. The technique is generally applicable to a large class of communications problems and provides a practical solution to many analytically intractable problems. The technique works equally well for discrete and continuous distributions and assumes no *a priori* knowledge of the distribution.

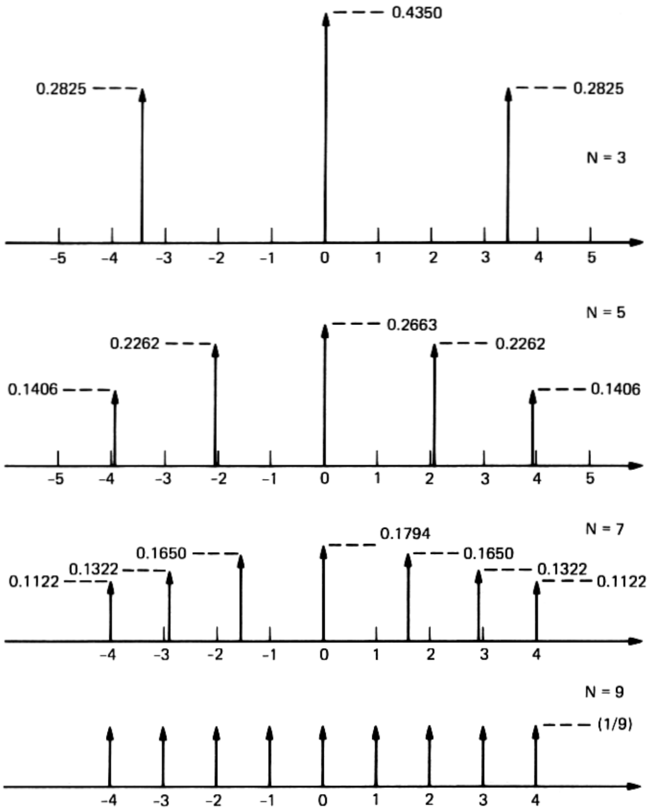

Fig. 6—GQR for discrete distribution.

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APPENDIX
Details of the GQR Algorithm

In this appendix, we outline the algorithm used to compute Gaussian Quadrature Rules. The procedure combines Gautchi's modified moment technique[14] with the Cholesky decomposition suggested by Martin et al.[15] The resulting algorithm has been implemented using double precision arithmetic and has proven stable and robust.

To compute the $2N$ unknowns $\{A_i\}_{i=1}^{N}$ and $\{t_i\}_{i=1}^{N}$, we first form the matrix of modified moments

$$\boldsymbol{M} = \begin{bmatrix} m_{1,1} & m_{1,2} \cdots & m_{1,N+1} \\ m_{2,1} & & \cdot \\ \vdots & & \vdots \\ m_{2,N+1} & \cdots & m_{N+1,N+1} \end{bmatrix}, \tag{21}$$

where m_{ij} is given by the inner product

$$\begin{aligned} m_{ij} &= (T_{i-1}, T_{j-1}) \\ &= \int_a^b T_{i-1}(x) T_{j-1}(x) dW(x) \quad i, j = 1, 2 \cdots N+1 \end{aligned} \tag{22}$$

and $\{T_i\}_{i=0}^{N}$ are the first $N + 1$ members of an arbitrary set of polynomials satisfying the recurrence relation

$$xT_j(x) = a_jT_{j+1}(x) + b_jT_j(x) + c_jT_{j-1}(x) \quad j = 0, 1, 2, \cdots N$$
$$T_{-1}(x) = 0, \qquad a_j \neq 0. \tag{23}$$

The orthogonal Tchebycheff polynomials determined by (23) constitute a convenient choice, with

$$a_0 = 1$$
$$a_j = c_j = 1/2 \quad j = 1, 2 \cdots$$
$$b_j = 0 \qquad j = 1, 2 \cdots. \tag{24}$$

The modified moments m_{ij} in (23) are simply linear combinations of the moments

$$\mu_j = \int_a^b x^j dW(x)$$

and can be simplified for the case of the Tchebycheff polynomials by using the relation

$$T_i(x)T_j(x) = 1/2\{T_{i+j}(x) + T_{i-j}(x)\} \quad i \geq j. \tag{25}$$

Thus, if we define

$$\nu_k = \int T_k(x)dW(x),$$

then

$$m_{ij} = 1/2\{\nu_{i+j-2} + \nu_{i-j}\} \quad i \geq j. \tag{26}$$

It is not necessary in theory for the $T_k(x)$ to be orthogonal. The formulation by Golub and Welsch[2] used the unmodified moments corresponding to $T_k(x) = x^k$ and, hence, $a_j = 1$, $b_j = 0$, and $c_j = 0$ for all j. As Gautchi shows,[14] the use of modified moments results in less sensitivity to computer roundoff errors.

We next form the tridiagonal matrix

$$\boldsymbol{J} = \begin{bmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \cdot & \cdot & \\ & \cdot & \cdot & \alpha_{N-1} & \beta_{N-1} \\ 0 & & & \beta_{N-1} & \alpha_N \end{bmatrix}, \tag{27}$$

where

$$\alpha_j = b_j + \frac{r_{j,j+1}}{r_{j,j}} a_j - \frac{r_{j-i,j}}{r_{j-1,j-1}} a_{j-1} \quad j = 1, 2 \cdots N$$

$$\beta_j = \frac{r_{j+1,j+1}}{r_{j,j}} a_j \qquad j = 1, 2 \cdots N - 1.$$

The r_{ij} are found from the relation

$$M = R^T R. \tag{28}$$

The matrix $\boldsymbol{R}$ is an upper triangular matrix and theoretically is positive definite if $\boldsymbol{M}$ is positive definite. In practice, however, $\boldsymbol{M}$ can be ill-conditioned and finite precision arithmetic will cause the matrix to appear singular.

The elements of $\boldsymbol{R}$ are related to the moment matrix $\boldsymbol{M}$ by the relations

$$r_{ii} = \left(m_{ii} - \sum_{k=1}^{i-1} r_{ki}^2\right)^{1/2}$$

$$r_{ij} = \left(m_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj}\right) \Big/ r_{ii} \quad i < j$$

$$i, j = 1, 2 \cdots N. \tag{29}$$

In practice, the computation of $\boldsymbol{R}$ from (29) will fail at relatively small values of N when the square root of a negative number is attempted.

A refined Cholesky decomposition[15] overcomes this problem by only requiring square roots to be computed at the end of the decomposition and not at each step as in (29). If we define $\boldsymbol{R}^*$ by the relation

$$R = R^* diag(r_{ii}),$$

then $\boldsymbol{R}^*$ will be a unit upper triangular matrix and

$$\begin{aligned} M &= R^T R \\ &= R^{*T} diag(r_{ii}^2) R^* \\ &= R^{*T} D R^*, \end{aligned} \tag{30}$$

where $\boldsymbol{D}$ is a positive diagonal matrix. Then, defining the auxiliary quantities

$$m_{ij}^* = r_{ij}^* d_j, \tag{31}$$

the following solution is obtained

$$m_{ij}^* = m_{ij} - \sum_{k=1}^{j-1} m_{ik}^* r_{jk}^* \quad j = 1, 2 \cdots i - 1$$

$$d_i = m_{ii} - \sum_{k=1}^{i=1} m_{ik}^* r_{ik}^*. \tag{32}$$

The advantage of the alternate decomposition is that square roots are not required until the final step, when the positive diagonal matrix

Table I—Comparison of three implementations

	Standard Cholesky	Alternate Cholesky	Alternate Cholesky Modified Moments
Gaussian Random Variable	17	60	60
Uniform Random Variable	13	40	38
Lognormal Random Variable	—	17	14
Symmetrized, Lognormal Random Variable	—	60	60

$\boldsymbol{D}$ in (30) is factored. Along with the modified moment procedure, the alternate Cholesky decomposition yields accurate results even for large values of N.

Several implementations of the GQR algorithm have been examined to elucidate the features that contribute to the reduction of computer roundoff errors. These include:

(*i*) Standard Cholesky
(*ii*) Alternate Cholesky
(*iii*) Alternate Cholesky with modified moments
(*iv*) All of the above using symmetrized moments.

Each approach was evaluated in double precision arithmetic.

The value of N at which the Cholesky decomposition fails was chosen as the measure of robustness for a variety of input probability density functions. Some of these results are tabulated in Table I. The standard Cholesky consistently had the poorest performance for all of the distributions considered. For symmetric distributions, the alternate Cholesky scheme provided a significant reduction of computer error. For the Gaussian distribution, the procedure was accurate for all $N \leq 60$, where 60 was the dimensionality limit imposed on the computer routine by storage requirements. The addition of the modified moment approach resulted in virtually no improvement relative to the alternate Cholesky implementation alone. None of the first three approaches proved satisfactory for nonsymmetric distributions (e.g., lognormal). The solution to this obstacle for one-sided distributions is to symmetrize the distribution according to (16), find the GQR estimate for the symmetrized distribution, and then obtain the desired distribution using (17). As we see in Table I, this renders the lognormal estimate as robust as the symmetric Gaussian distribution.

The final step in obtaining the nodes and weights involves finding the eigenvalues and eigenvectors of the matrix $\boldsymbol{J}$ in (27). The eigenvector $\underline{q}_j$ corresponding to the eigenvalue t_j is found from the equation

$$\boldsymbol{J}\underline{q}_j = t_j\underline{q}_j \quad j = 1, 2, \cdots N. \tag{33}$$

The eigenvalues $\{t_j\}_{i=l}^{N}$ are the nodes of the GQR and the positive weights are given by

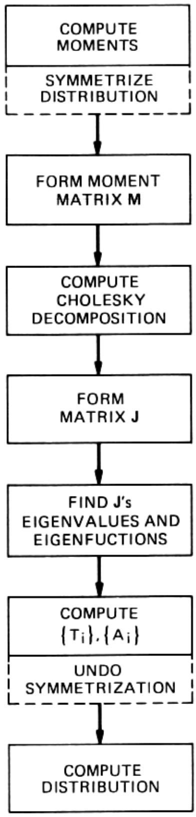

Fig. 7—Flowchart of GQR algorithm.

$$A_j = q_{1j}^2 \mu_0, \tag{34}$$

where

$$\underline{q}_j^T = (q_{1j}, q_{2j} \cdots q_{N,j}).$$

A flowchart of the steps used to compute GQR is shown in Fig. 7.