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On the Average Product of Gauss-Markov Variables

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Let x_i be members of a stationary sequence of zero mean gaussian random variables having correlations $Ex_ix_j = \sigma^2\rho^{|i-j|}$, $0 < \rho < 1$, $\sigma > 0$. We address the behavior of the averaged product $q_m(\rho, \sigma) \equiv Ex_1x_2 \cdots x_{2m-1}x_{2m}$ as m becomes large. Our principal result when $\sigma^2 = 1$ is that this average approaches zero (infinity) as ρ is less (greater) than the critical value $\rho_c = 0.563007169 \ldots$. To obtain this we introduce a linear recurrence for the $q_m \cdot (\rho, \sigma)$, and then continue generating an entire sequence of recurrences, where the $(n + 1)$-st relation is a recurrence for the coefficients that appear in the nth relation. This leads to a new, simple continued fraction representation for the generating function of the $q_m(\rho, \sigma)$. The related problem with $\bar{q}_m(\rho, \sigma) = E|x_1 \cdots x_m|$ is studied via integral equations and is shown to possess a smaller critical correlation value.

I. INTRODUCTION

The problem that we consider in this paper is as follows: Let $\{x_i\}_1^\infty$ be a stationary sequence of zero mean, gaussian random variables with covariances

$$\rho_{ij} \equiv Ex_ix_j = \sigma^2\rho^{|i-j|},\ 0 < \rho < 1,\ \sigma > 0; \qquad i, j = 1, 2, \cdots, \tag{1}$$

where $E(\cdot)$ denotes mathematical expectation. What is the behavior of

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$$q_m(\rho, \sigma) \equiv Ex_1x_2 \cdots x_{2m-1}x_{2m} \tag{2}$$

as m becomes large?

In other words, the product in (2) is formed from samples of a gauss-markov process that are taken at regular intervals. Only an even number of samples is considered in (2) since an odd number would result in a zero average.

Originally, the problem was conceived as a simple model for averages of multiplicative structures having infinite memory between the factors of the product. Such products arise in the analysis of learning curves for many adaptive systems, and for these problems one encounters products whose factors are noncommuting matrices. We felt that the analysis of a simple problem, such as that described above, would serve as a valuable guide to what results might be achievable for more realistic situations. However as one may readily imagine, as soon as the problem described in (1) and (2) was written down it became of interest in its own right, consisting as it does of a simple question about long familiar quantities.

Our principal result is that for large m the behavior of the average product $q_m(\rho, \sigma)$ in (2) depends on the relationship of ρ to a critical value, $\rho_c = \rho_c(\sigma)$. If $\rho < \rho_c$, then $q_m(\rho, \sigma)$ will approach zero exponentially fast; if $\rho > \rho_c$, $q_m(\rho, \sigma)$ approaches infinity exponentially fast; finally, if $\rho = \rho_c$, $q_m(\rho, \sigma) \to q_\infty(\sigma)$. We find for $\sigma = 1$, $\rho_c(1) = 0.563007169391816 \cdots$, and $q_\infty(1) = 0.50900853 \ldots$. A plot of $\rho_c(\sigma)$ is given in Fig. 1. All of these results were obtained from a continued fraction representation for the generating function

$$Q(z, \rho, \sigma) = \sum_{m=0}^{\infty} q_m(\rho, \sigma) z^m. \tag{3}$$

Since $q_m(\rho, \sigma) = \sigma^{2m} q_m(\rho, 1)$, we have

$$Q(z, \rho, \sigma) = Q(z\sigma^2, \rho, 1), \tag{4}$$

so it is without loss of generality that we will set $\sigma = 1$, $Q(z, \rho) = Q(z, \rho, 1)$, and $q_m(\rho) = q_m(\rho, 1)$. By introducing a sequence of generating functions, we show in Section II that

$$Q(z, \rho) = \cfrac{1}{1 - \cfrac{\rho z}{1 - \cfrac{2\rho^3 z}{1 - \cfrac{3\rho^5 z}{1 - \ddots}}}} \tag{5}$$

The value $\rho_c^{(\sigma)}$ is then the smallest ρ for which $Q(\sigma^2, \rho) = \infty$, while the value $q_\infty(\sigma)$ is the limit as $z \rightarrow 1$ of $(1 - z)Q(z\sigma^2, \rho_c)$.

Since methods are as interesting as results, Section III presents another approach involving integral equations for discussing the $q_m(\rho)$ behavior. Although this method is not rigorously justified for the present problem due to a non-hermitian kernel, it is applicable to a

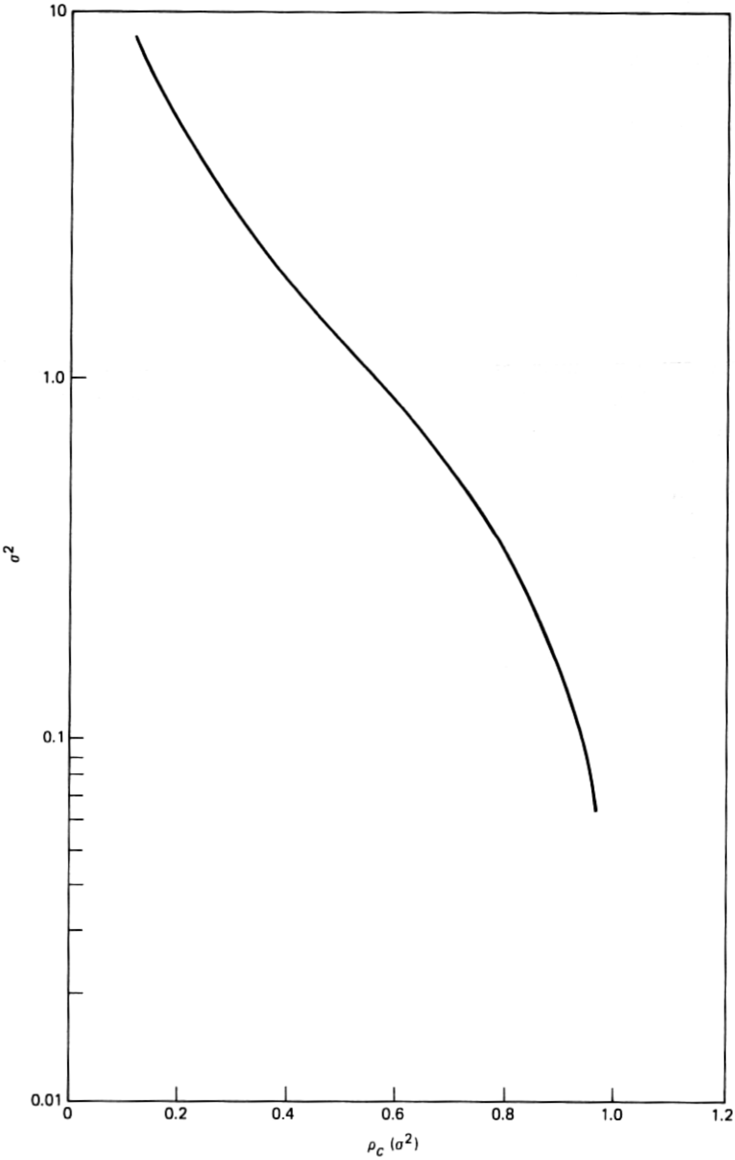

Fig. 1—Critical correlation value $\rho_c(\sigma^2)$ vs. variance σ^2.

related problem, the behavior of $E|x_1 \cdots x_m|$ as $m \to \infty$ [still assuming (1)]. Using the integral equation we show that $\bar{\rho}_c$, the critical value of ρ for this new problem, is strictly smaller than the ρ_c defined above. This is of interest since it shows that the behavior of $q_m(\rho)$ is determined both by how large $|x_1 \cdots x_{2m}|$ is on the average, and by the extent of cancellation between positive and negative values of q_m.

Although we do not give the details here, it is not difficult to show that for all $\rho < 1$, $q_m(\rho)$ approaches zero with probability one as m becomes large.

II. LINEAR RECURRENCES AND GENERATING FUNCTIONS

Given $2m$ zero-mean jointly gaussian random variables x_i of unit variance and correlations $Ex_ix_j = \rho_{ij}$, then a known formula[1] states that

$$Ex_1 \cdots x_{2m} = \sum_{\substack{\text{all}\\ \text{pairs}}} \rho_{i_1 i_2}\rho_{i_3 i_4} \cdots \rho_{i_{2m-1} i_{2m}}, \tag{6}$$

where the unordered set $\{i_1, \cdots, i_{2m}\}$ is equal to the unordered set $\{1, 2, \cdots, 2m\}$. The sum in (6) is over all distinct, unordered pairs of subscripts. That is, we do not count twice terms which differ only by interchanging the values within one or more subscript pairs, nor do we count twice terms which differ only by permuting subscript pairs. Thus there are $(2m)!/(2^m m!)$ terms in the sum (6).

If we denote permutations of $2m$ objects by $\sigma(i)$: $i \to \sigma(i)$, $i = 1, 2, \cdots, 2m$, then a succinct way of writing (6) when (1) holds is

$$q_m(\rho) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \rho^{\sum_{j=1}^{m} |\sigma(2j) - \sigma(2j-1)|}, \tag{7}$$

the sum in (7) being over all $(2m)!$ permutations of S_{2m}, the group of permutations of $2m$ symbols. Formula (7) shows immediately that $q_m(\rho) > 0$ if $\rho > 0$.

Now define $q_0(\rho) = 1$ and write

$$q_m(\rho) = \sum_{s=1}^{m} b_s(\rho) q_{m-s}(\rho), \qquad m = 1, 2, \cdots. \tag{8}$$

We evaluate a few of the $b_s(\rho)$, writing for convenience $b_i(\rho) = b_i$, $q_i(\rho) = q_i$. The evaluation is done from (8) by explicitly evaluating the $q_m(\rho)$ as needed. A partial list of $b_i(\rho)$ follows:

$$b_1 = \rho$$

$$b_2 = 2\rho^4$$

$$b_3 = 4\rho^7 + 6\rho^9$$

$$b_4 = 8\rho^{10} + 24\rho^{12} + 18\rho^{14} + 24\rho^{16}$$

$$b_5 = 16\rho^{13} + 72\rho^{15} + 108\rho^{17} + 150\rho^{19} + 144\rho^{21} + 96\rho^{23} + 120\rho^{25}. \tag{9}$$

Equation (9) suggests the possibility that, for small ρ, only a few terms in (8) would need to be kept for an accurate description of $q_m(\rho)$. For example, keeping only one term yields

$$q_m = \rho q_{m-1}, \tag{10}$$

or $q_m = \rho^m$. Since $Ex_1x_2 = \rho$, this approximation corresponds to treating the successive pairs of gaussian variables which determine $q_m(\rho)$, via (2), as independent. The next step after (10) would be to write

$$q_m = b_1 q_{m-1} + b_2 q_{m-2}. \tag{11}$$

This equation, involving b_2 as well, would be a correction to the "independence assumption," but one involving only up to fourth-order correlations, since, from (8) the highest average appearing in b_2 is $E(x_1x_2x_3x_4)$. Further corrections are obtained by including more terms of (8), with higher order correlations entering.†

Assuming the $b_i(\rho)$ to be known, the natural procedure would be to "solve" (8) using generating functions. We define these as follows: if $y_0, y_1, y_2, \cdots$ is a bounded sequence of numbers, then the generating function, $Y(z)$, of the sequence is defined for complex z, $|z| < 1$, by

$$Y(z) = \sum_{i=0}^{\infty} y_i z^i. \tag{12}$$

Given $Y(z)$, the y_i are, in principle, uniquely determined. We assume that the reader is familiar with the use of generating functions. If not, consult Chapters XI and XIII of Feller.[2]

We define $b_0(\rho) = 0$, $q_0(\rho) = 1$, and call the generating functions of the $b_i(\rho)$, and $q_i(\rho)$ sequences $B(z;\rho)$ and $Q(z;\rho)$, respectively. The ρ dependence is explicitly indicated.

If we multiply (8) by z^m and sum from $m = 1$ to ∞ (treating $q_m = 0$, $m < 0$ and $b_m = 0$, $m < 0$), we obtain the basic relation

$$Q(z;\rho) = \frac{1}{1 - B(z;\rho)}. \tag{13}$$

Equation (13) thus allows us to determine, in principle, the q_m from the b_m. In particular, we have

† The above interpretation prompts us to advocate consideration of the ideas represented by (8) for analyzing more complex multiplicative structures, particularly when connections to some sort of independence approximation are a natural thing to seek.

$$\sum_{m=1}^{\infty} q_m(\rho) = Q(1;\, \rho) = \frac{1}{1 - B(1;\, \rho)}, \tag{14}$$

and the critical value ρ_c will be given by the equation

$$B(1;\, \rho_c) = \sum_{1}^{\infty} b_k(\rho_c) = 1. \tag{15}$$

Although we could work with the $b_m(\rho)$ themselves, a more convenient approach for finding ρ_c numerically is to set up a continued fraction representation for the generating functions $Q(z;\, \rho)$, or equivalently, $B(z;\, \rho)$. It is this approach that we follow now.

Recall (8) defining $b_s(\rho)$. Since these b_s coefficients are a numerical sequence themselves, we can use the same reasoning that took us from the q_k to the b_s and use it to suggest going from the b_s to a new set of coefficients, $b_k^{(2)}$, via the following recurrence

$$b_k(\rho) = \sum_{s=1}^{k} b_s^{(2)}(\rho) b_{k-s}(\rho), \qquad k = 2, 3, \cdots, \tag{16}$$

where we define $b_0(\rho) = 0$. The recurrence (16) yields

$$B(z;\, \rho) = \frac{b_1(\rho) z}{1 - B^{(2)}(z;\, \rho)}, \tag{17}$$

$B^{(2)}(z;\, \rho)$ being the generating function for the $b_k^{(2)}(\rho)$. To continue this procedure with a uniform notation, we define

$$\begin{aligned} b_s^{(1)}(\rho) &= b_s(\rho) \\ b_0^{(m)}(\rho) &= 0, \qquad m = 1, 2, \cdots \end{aligned} \tag{18}$$

and write

$$b_k^{(m)}(\rho) = \sum_{s=1}^{k} b_s^{(m+1)}(\rho) b_{k-s}^{(m)}(\rho), \qquad \begin{matrix} m = 1, 2, \cdots \\ k = 2, 3, \cdots \end{matrix}. \tag{19}$$

The corresponding sequence of generating functions are related by

$$B^{(m)}(z;\, \rho) = \frac{b_1^{(m)}(\rho) z}{1 - B^{(m+1)}(z;\, \rho)}. \tag{20}$$

We use this repeatedly in (13) and obtain the continued fraction representation†

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† The fact that this continued fraction does not terminate implies that $Q(z;\, \rho)$ is not a rational function of z, and thus one cannot find a (finite-order) difference equation for the $q_m(\rho)$. See Ref. 3, Theorem 99.1, p. 400.

$$Q(z, \rho) = \cfrac{1}{1 - \cfrac{b_1^{(1)} z}{1 - \cfrac{b_1^{(2)} z}{1 - \cfrac{b_1^{(3)} z}{1 - \ddots}}}} \tag{21}$$

In (21) we have, for simplicity, written $b_1^{(m)}(\rho) = b_1^{(m)}$.

A relation which will be used later to aid in finding the $b_1^{(m)}$ follows by setting $k = 2$ in (19), to obtain

$$b_1^{(m+1)}(\rho) = \frac{b_2^{(m)}(\rho)}{b_1^{(m)}(\rho)}. \tag{22}$$

We can calculate some of the $b_1^{(m)}(\rho)$ using the partial list of the $b_k(\rho)$ given in (9) to derive several $b_s^{(m)}(\rho)$ from (19). Using (22) we then obtain

$$\begin{aligned} b_1^{(1)} &= \rho \\ b_1^{(2)} &= 2\rho^3 \\ b_1^{(3)} &= 3\rho^5 \\ b_1^{(4)} &= 4\rho^7 \\ b_1^{(5)} &= 5\rho^9. \end{aligned} \tag{23}$$

The obvious guess that

$$b_1^{(m)} = m\rho^{2m-1}, \qquad m = 1, 2, \cdots \tag{24}$$

follows from a direct proof of the continued fraction given in the appendix. Assuming (24) to hold yields the simple representation

$$Q(z; \rho) = \cfrac{1}{1 - \cfrac{\rho z}{1 - \cfrac{2\rho^3 z}{1 - \cfrac{3\rho^5 z}{1 - \ddots}}}} \tag{25}$$

The accurate numerical value

$$\rho_c = 0.563007169391816 \cdots \tag{26}$$

was obtained by using this representation along with (14) and (15).

When $\rho = \rho_c$,

$$q_\infty = \lim_{k\to\infty} q_k(\rho_c) = \frac{1}{\sum_1^\infty kb_k(\rho_c)}. \tag{27}$$

Using the computed value of ρ_c and the definition of the $b_k(\rho)$, we found numerically that

$$q_\infty = 0.50900853 \cdots. \tag{28}$$

It was quite surprising to us that $Q(z, \rho)$ turned out to be a new, but simple, continued fraction.

III. INTEGRAL EQUATION METHOD

The purpose of this section is to introduce the integral equation method and to show that $\bar{\rho}_c < \rho_c$, where $\bar{\rho}_c$ is the critical correlation value for the related problem involving $E|x_1 \cdots x_n|$.

We begin by developing an expression for $Ex_1 \cdots x_n$. We have, from the Markov property of the x_i sequence,

$$Ex_1 \cdots x_n = \int \cdots \int x_n p(x_n | x_{n-1}) \cdots x_1 p(x_1 | x_0)\phi(x_0)dx_0 \cdots dx_n, \tag{29}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \tag{30}$$

is the standard normal density and

$$p(y|x) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp[-(y-\rho x)^2/2(1-\rho^2)] \tag{31}$$

is the generic form of the conditional densities occurring in (29). Define a kernel $K(x, y)$ by

$$K(x, y) = yp(y|x),$$

$$Kf(x) = \int_{-\infty}^{\infty} K(x, y)f(y)dy.$$

Then (29) may be written in the inner product notation of Hilbert Space

$$Ex_1 \cdots x_n = (K^n \underline{1}, \phi), \tag{32}$$

where ϕ is the normal density (30), $\underline{1}$ is the unit constant function, and K^n is the nth iterated kernel. Now assume, heuristically, that K^n has the usual expansion

$$K^n(x, y) = \sum_{j=1}^{\infty} \lambda_j^n \psi_j(x) \psi_j(y) \tag{33}$$

in terms of eigenfunctions $\psi_j(x)$ and eigenvalues λ_j of K. Then $Ex_1 \cdots x_n$ would remain bounded, if, and only if, the largest eigenvalue $\lambda_1 = \lambda_1(\rho)$ is less or equal to one; thus $\lambda_1(\rho_c) = 1$ would determine ρ_c. Unfortunately there is no general eigenexpansion theory available for K since it is not symmetric and is not symmetrizable.

Fortunately symmetry holds for the integral equation method when one expresses $E|x_1 \cdots x_n|$ via kernels. Define, in analogy to (32),

$$\bar{K}(x, y) = |y| p(y|x). \tag{34}$$

If we further define

$$J(x, y) = \frac{h(x)}{h(y)} \bar{K}(x, y), \tag{35}$$

where

$$h(x) = \sqrt{|x|} \exp(-x^2/4), \tag{36}$$

then

$$J(x, y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \sqrt{|xy|} \exp\left[-\frac{1}{4} \frac{1+\rho^2}{1-\rho^2} (x^2 + y^2)\right] \cdot \exp\left[\frac{\rho x y}{1-\rho^2}\right] \tag{37}$$

is a symmetric kernel.

As in (29),

$$\begin{aligned} E|x_1 \cdots x_n| &= \int \cdots \int \bar{K}(x_{n-1}, x_n) \bar{K}(x_{n-2}, x_{n-1}) \cdots \bar{K}(x_0, x_1)\, \phi(x_0) dx_0 \cdots dx_n \\ &= \int \cdots \int dx_0 \cdots dx_n\, J(x_0, x_1) \cdots J(x_{n-1}, x_n) \frac{h(x_n)}{h(x_0)} \phi(x_0) \\ &= \left(J^n h, \frac{\phi}{h}\right). \end{aligned} \tag{38}$$

Since J is symmetric and square-integrable, it is a Hilbert-Schmidt kernel and so has a discrete spectrum. Further its maximum eigenvalue, λ, is given by

$$\lambda = \sup_f \frac{(Jf, f)}{(f, f)}. \tag{39}$$

Since $J(x, y) \geq 0$, we see that the maximum eigenfunction $g = g(x)$ is nonnegative and $\lambda > 0$ as well. Further since h and ϕ/h are nonnegative, $(h, g) > 0$ and $(\phi/h, g) > 0$ so that $E|x_1 \cdots x_n| = (J^n h, \phi/h) \rightarrow \infty$ if and only if $\lambda > 1$.

Define f_α by

$$f_\alpha(x) = \sqrt{|x|}\exp(-\alpha x^2/4), \tag{40}$$

and note that from (39),

$$\lambda > (Jf_\alpha, f_\alpha)/(f_\alpha, f_\alpha). \tag{41}$$

Now

$$(f_\alpha, f_\alpha) = \int |x|\exp(-\alpha x^2/2)dx = 2/\alpha \tag{42}$$

and

$$(Jf_\alpha, f_\alpha) = \frac{1}{\sqrt{2\pi(1-\rho^2)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|xy| \cdot \exp\left[-\frac{c}{2}(x^2+y^2) + \frac{\rho xy}{1-\rho^2}\right]dxdy, \tag{43}$$

where

$$c = \frac{1}{2}\left(\frac{1+\rho^2}{1-\rho^2} + \alpha\right). \tag{44}$$

Set $y = xu$ and integrate over x to obtain

$$(Jf_\alpha, f_\alpha) = \frac{4}{\sqrt{2\pi(1-\rho^2)}}\int_{-\infty}^{\infty}\frac{|u|du}{(c(1+u^2)-\beta u)^2},$$

$$\beta = \frac{2\rho}{1-\rho^2}. \tag{45}$$

Using Ref. 4 (p. 68, 2.175) we evaluate the last integral as

$$(Jf_\alpha, f_\alpha) = \frac{4}{\sqrt{2\pi(1-\rho^2)}}\left[\frac{4}{\Delta} + \frac{4\beta}{\Delta^{3/2}}\tan^{-1}\frac{\beta}{\sqrt{\Delta}}\right],$$

$$(\Delta = 4c^2 - \beta^2. \quad (46)$$

Setting $\alpha = 0.5$, $\rho = 0.55$, we find that $c = 1.18369$, $\beta = 1.57706$, $\Delta = 3.11738$, $(Jf_\alpha, f_\alpha) = 4.0824$, $(f_\alpha, f_\alpha) = 4$, so that $\lambda > 1.012$. Thus for $\rho = 0.55$, $E|X_1 \cdots X_n| \to \infty$, and $\bar{\rho}_c < 0.55$. We have seen that $\rho_c > 0.563$ so the claim is proven.

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APPENDIX

Combinatorial Derivation of Continued Fraction

In this appendix we give a direct combinatorial proof of the continued fraction representation (25) of $Q(z, \rho)$. This derivation is complete in itself, but we preferred the method of the text for showing where the continued fraction comes from. Our starting point is the formula (7). Let us define, for $\sigma \in S_{2m}$,

$$V(\sigma) = \sum_{i=1}^{m} |\sigma(2i) - \sigma(2i-1)|. \quad (47)$$

For $1 \leqslant k \leqslant m$, let

$$S(m, k) = \{\sigma \in S_{2m}: \sigma(2m) = 2m, \sigma(2m-2) = 2m-1, \sigma(2m-4) = 2m-2, \cdots, \sigma(2m-2k+2) = 2m-k+1\}. \quad (48)$$

For $k = 0$, we adopt the convention that $S(m, 0) = S_{2m}$. We also define

$$u(m, k) = \frac{1}{2^{m-k}(m-k)!} \sum_{\sigma \in S(m,k)} \rho^{V(s)}, \quad (49)$$

so that $u(m, 0) = q_m$. (We take $u(0, 0) = q_0 = 1$, and $u(m, k) = 0$ for $k < 0$ and $k > m$.) Our key result is:

Lemma. If $m \geqslant 1$, $k \geqslant 0$, then

$$u(m, k) = k\rho^{2k-1}u(m-1, k-1) + u(m, k+1). \tag{50}$$

Proof. We will prove this for $1 \leqslant k \leqslant m-1$, as the other cases are easy. Let

$$S' = \{\sigma \in S(m, k): 2m-k \in \{\sigma(2m-1), \sigma(2m-3), \cdots, \sigma(2m-2k+1)\}\},$$
$$S'' = S(m, k) - S'. \tag{51}$$

If $\sigma \in S''$, we construct a permutation $\sigma^* \in S(m, k+1)$ by changing the action of σ on four letters in such a way that $V(\sigma) = V(\sigma^*)$ and $\sigma^*(2m-2k) = 2m-k$. To define σ^* precisely, let p and r be such that $\{r, 2m-k\} = \{\sigma(2p-1), \sigma(2p)\}$. Then, if we associate to σ the vector $A(\sigma) = (\sigma(1), \sigma(2), \cdots, \sigma(2m))$, the vector $A(\sigma^*)$ is obtained from $A(\sigma)$ by interchanging the pairs $\{\sigma(2m-2k-1), \sigma(2m-k)\}$ and $\{r, 2m-k\}$ so as to keep the same ordering in the first pair, but possibly reversing it in the second, so as to have $\sigma^*(2m-2k) = 2m-k$. As an example, if $m = 5$, $k = 3$, and $A(\sigma) = (7, 2, 4, 1, 6, 8, 3, 9, 5, 10)$, then $A(\sigma^*) = (4, 1, 2, 7, 6, 8, 3, 9, 5, 10)$. It is clear that $\sigma^* \in S(m, k+1)$ and $V(\sigma^*) = V(\sigma)$. Moreover, every $\tau \in S(m, k+1)$ can be represented in exactly $2(m-k)$ ways as $\tau = \sigma^*$, $\sigma^* \in S''$. Therefore,

$$\frac{1}{2^{m-k}(m-k)!}\sum_{\sigma \in S''} \rho^{V(\sigma)} = u(m, k+1). \tag{52}$$

Suppose now that $\sigma \in S'$. Then $2m-k = \sigma(2m-2r+1)$ for some r, $1 \leqslant r \leqslant k$. We now define a permutation $\sigma' \in S(m-1, k-1)$ as follows: In $A(\sigma)$, delete $a = \sigma(2m-2r+1)(= 2m-k)$ and $b = \sigma \cdot (2m-2r+2)$ and reduce the remaining entries that are between a and b by 1, and those that are larger than $\max(a, b) = a$ by 2. As an example, if $m = 5$, $k = 3$, and $A(\sigma) = (2, 1, 6, 3, 5, 8, 7, 9, 4, 10)$, then $A(\sigma') = (2, 1, 6, 3, 5, 7, 4, 8)$. The resulting vector clearly equals $A(\sigma')$ for some $\sigma' \in S(m-1, k-1)$, and each $\tau \in S(m-1, k-1)$ has exactly k such representations. Further, $V(\sigma)$ equals the sum of (i) $V(\sigma')$, (ii) $a-b$ for the pair that was dropped, (iii) 2 for each of the $r-1$ pairs $(\sigma(2m-2j+1), \sigma(2m-2j+2))$ for $1 \leqslant j \leqslant r-1$, since in each such pair $\sigma(2m-2j+2) > a$, $\sigma(2m-2j+1) < b$, and finally (iv) 1 for each of the $k-r$ pairs $(\sigma(2m-2j+1), \sigma(2m-2j+2))$, $r + \leqslant j \leqslant k$, since in each of them $\sigma(2m-j+1) < b$, $b < \sigma(2m-2j+2) < a$. Hence,

$$V(\sigma) = V(\sigma') + a - b + 2(r-1) + k - r. \tag{53}$$

But $a = 2m-k$ and $b = \sigma(2m-2r+2) = 2m-r+1$ from the definitions of $S(m, k)$, so

$$V(\sigma) = V(\sigma') + 2k - 1. \tag{54}$$

Hence, we have

$$\frac{1}{2^{m-k}(m-k)!} \sum_{\sigma \in S'} \rho^{V(\sigma)} = k\rho^{2k-1}u(m-1, k-1), \tag{55}$$

which proves the lemma.

We now can use the recurrence of the Lemma to derive the continued fraction expansion of the generating function. Let

$$f_k = f_k(z) = \sum_{m=0}^{\infty} u(m, k)z^m, \qquad k = 0, 1, \cdots,$$

which for the moment we regard as formal power series in z. Then the Lemma gives us

$$f_1 = f_0 - 1, \tag{56}$$

and for $k \geqslant 2$,

$$\begin{aligned} f_k &= \sum_m u(m, k-1)z^m - (k-1)\rho^{2k-3} \sum_m u(m-1, k-2)z^m \\ &= f_{k-1} - (k-1)\rho^{2k-3} z f_{k-2}. \end{aligned} \tag{57}$$

Relations (56) and (57) show that for $k \geqslant 0$,

$$f_k = s_k f_0 - r_k, \tag{58}$$

where $s_0 = s_1 = 1$, $r_0 = 0$, $r_1 = 1$, and for $k \geqslant 2$ both s_k and r_k satisfy the recurrence

$$x_k = x_{k-1} - (k-1)\rho^{2k-3} z x_{k-2}.$$

Hence the quotients r_k/s_k are the partial quotients of the continued fraction $R(z, \rho)$ on the right side of (25), and s_k and r_k converge as $k \to \infty$ to power series (in z) $s(z, \rho)$ and $r(z, \rho)$, respectively, for which

$$R(z, \rho) = \frac{r(z, \rho)}{s(z, \rho)}. \tag{59}$$

On the other hand, since f_k starts with a term involving z_k, we conclude that f_k converges to 0 in the ring of formal power series as $k \to \infty$. Therefore, from (58),

$$f_0 = \frac{r(z, \rho)}{s(z, \rho)} = R(z, \rho). \tag{60}$$

Since $f_0 = Q(z, \rho)$, we obtain the relation (25), at least in the ring of formal power series in z. However, the continued fraction (25) is clearly a meromorphic function of z for ρ fixed, $0 < \rho < 1$, and it is analytic at 0. Hence (25) holds as an equality among meromorphic functions, and we can obtain from this the exponential decrease of the $q_m(\rho)$ for $\rho < \rho_c$ and the exponential increase for $\rho > \rho_c$.

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